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Abstract

Strategy-proof and envy-free random assignment

by Christian Basteck and Lars Ehlers*

We study the random assignment of indivisible objects among a set of agents with strict preferences. We show that there exists no mechanism which is unanimous, strategy-proof and envy-free. Weakening the first requirement to q-unanimity – i.e., when every agent ranks a different object at the top, then each agent shall receive his most-preferred object with probability of at least q – we show that a mechanism satisfying strategy-proofness, envy-freeness and ex-post weak non-wastefulness can be q-unanimous only for $q \leq 2/n$ (where n is the number of agents). To demonstrate that this bound is tight, we introduce a new mechanism, Random-Dictatorship-cum-Equal-Division (RDcED), and show that it achieves this maximal bound when all objects are acceptable. In addition, for three agents, RDcED is characterized by the first three properties and ex-post weak efficiency. If objects may be unacceptable, strategy-proofness and envy-freeness are jointly incompatible even with ex-post weak non-wastefulness..

Keywords: random assignment, strategy-proofness, envy-freeness, q-unanimity.

JEL classification: D63, D70

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1. Introduction

Consider the problem of assigning indivisible objects among a set of agents – each agent is to receive at most one and we assume they have strict preferences over the set of objects. Further, while objects' characteristics may include a fixed monetary payment, there are no additional transfers. Problems like this arise in many real-life applications such as on-campus housing (where rents are fixed), organ allocation, school choice with ties in applicants' priorities, etc. Whenever several agents would like to consume the same object, the indivisibility of objects, together with the absence of any compensating transfers, will render any deterministic assignment unfair. This is the main reason for implementing random assignments in such contexts.

Since agents' preferences are private information, the design of random assignment mechanisms has to provide incentives to report them truthfully (as otherwise the assignment is based on false preferences). Moreover, in many applications, the resulting assignments should be based on agents' (ordinal) rankings of objects, rather than on their preferences over all possible lotteries as the elicitation of the latter is difficult in practice – for example, school choice programs will typically ask applicants to provide a list of schools, ranked from most- to least-preferred.

Strategy-proofness makes truthful reporting a dominant strategy and thus should ensure that agents truthfully reveal their ordinal preferences over objects for any underlying utility representation of preferences. Unfortunately, the literature on random assignment mechanisms contains several impossibility results as soon as strategy-proofness and equal-treatment-of-equals, as a minimal fairness requirement, are married with different ex-ante/ex-post notions of efficiency.[1] In some sense, these efficiency notions are hence "too strong". In addition, equal-treatment-of-equals may be considered "too weak" a notion of fairness as it only constrains random assignments in rare cases where agents' preferences are identical, which seems contrived given that fairness concerns were the principal reason to consider random assignments in the first place. Our paper keeps the strategy-proofness requirement, strengthens equal-treatment-of-equals to envy-freeness and explores the efficiency frontier given these two constraints. From a practical point of view, this allows to answer the question whether losses in efficiency are mild enough to allow to insist on strategy-proofness and envy-freeness. For example, in the related problem of school choice with priorities, the well known Deferred Acceptance mechanism is strategy-proof and justified-envy-free,[2] and is hence widely used despite being inefficient.

First, we marry strategy-proofness and envy-freeness with arguably one of the weakest well known efficiency requirements, namely unanimity. In our setting it

[1] Throughout 'ex-ante' is to be understood as before realizing the final deterministic assignment; this corresponds to the term 'interim' used in mechanism design outside of the literature on random assignments.

[2] That is, no agent envies another agent with lower priority.

requires that if all agents rank different objects first, then each agent shall receive his most-preferred object with probability one – in other words, whenever there exists a unique efficient assignment, then this assignment is chosen for sure. Unfortunately, we find this requirement to yield another impossibility (together with strategy-proofness and envy-freeness). Given this, we introduce a quantitative measure of how much unanimity is respected: q-unanimity means that in any such situation every agent receives his most-preferred object with probability of at least q. Of course, 0-unanimity is satisfied by any mechanism and by lowering q from one to zero we obtain a possibility together with strategy-proofness and envy-freeness. The important question is to determine the exact bound. We show that for two or more agents this bound is equal to $\frac{2}{n}$, where n denotes the number of agents – no mechanism may be q-unanimous for any q larger than $\frac{2}{n}$. We also introduce object-unanimity whereby a certain object shall be assigned to a specific agent who ranks this object first while all other agents rank it last. We show that the impossibility pertains when q-unanimity is replaced with q-object-unanimity (whereby the specific agent shall receive his most preferred object with probability of at least q).

To demonstrate that this bound can be achieved, we introduce a new mechanism, called Random-Dictatorship-cum-Equal-Division mechanism (RDcED). In this mechanism, any agent is chosen with equal probability to chose as dictator and receive his most-preferred object, while all other objects are assigned uniformly (at random) among the remaining agents. For three agents we show that RDcED is the unique mechanism which is ex-post weakly non-wasteful, ex-post weakly efficient, strategy-proof and envy-free on the domain where all objects are acceptable. Hence, RDcED is characterized by a natural set of properties for three agents. RDcED satisfies $\frac{2}{3}$-unanimity for three agents (as any agent is chosen with probability $\frac{1}{3}$ to be the dictator and when another agent is the dictator, the agent receives his most-preferred object with probability $\frac{1}{2}$); for an arbitrary number n of agents, RDcED satisfies $\frac{2}{n}$-unanimity and hence achieves the maximal bound for q-unanimity among all mechanism that are ex-post weakly non-wasteful, strategy-proof and envy-free. The same is true for q-object-unanimity.

When agents may consider objects unacceptable, our impossibility result becomes more severe. Strategy-proofness and envy-freeness are then jointly incompatible with even ex-post weak non-wastefulness – that is, we have to accept situations where, ex-post, an agent remains unassigned even though there exists an unassigned object that is acceptable to that agent. Only after further weakening ex-post weak non-wastefulness by restricting it to hold for profiles where there exists a unique non-wasteful assignment of objects do we arrive at a possibility result.

Finally, we show that by allowing waste on the domain of acceptable objects, one can increase the bound for q-unanimity beyond $\frac{2}{3}$ for three agents. More precisely, we construct a mechanism for three agents which (i) assigns any agent no object with probability $\frac{1}{6}$ and (ii) satisfies strategy-proofness, envy-freeness and $\frac{5}{6}$-unanimity. However, gains in efficiency (at least among agents that are assigned an object) by allowing for waste are limited – we show that the maximal level of q-unanimity achievable by allowing for waste is monotonically decreasing in the number of agents and that for three agents both q-unanimity and q-object-unanimity can be satisfied with at most $q = \frac{17}{18}$ in the class of strategy-proof and envy-free mechanisms.

Below we discuss the related literature in detail . The main starting point is the impossibility of strategy-proofness, envy-freeness and ex-ante efficiency. Bogomolnaia and Moulin [2001] show that this remains unchanged when envy-freeness is weakened to equal-treatment-of-equals. Furthermore, they introduce the probabilistic serial (PS) mechanism and show that it is envy-free and ex-ante efficient (hence necessarily violates strategy-proofness).[3] Nesterov [2017] shows that the impossibility persists when ex-ante efficiency is weakened to ex-post efficiency.[4] In closely related ongoing work, overlapping with our first result, Shende and Purohit [2020] show independently that strategy-proofness and envy-freeness are incompatible with unanimity (which they refer to as contention-free efficiency). After observing this impossibility, the authors move on by restricting attention to so-called (Balanced) Pairwise Exchange Mechanisms. They show in this limited class of mechanisms the equivalence strategy-proofness and envy-freeness, and analyze them in further detail. In contrast, we keep the set of mechanisms unrestricted throughout and determine the possibility frontier (in terms of unanimity).[5] Recently, Mennle and Seuken [2021] decomposed strategy-proofness into three properties which they call swap-monotonicity, upper invariance and lower invariance.

In terms of possibility results, it is known that the random serial dictatorship (RSD, also known as random priority) mechanism satisfies strategy-proofness, equal-treatment-of-equals and ex-post efficiency – i.e., weakening *both* envy-freeness and ex-ante efficiency results in a possibility. Our contribution is to keep envy-freeness (since fairness understood as equity is the principal reason for implementing a random assignment) and strategy-proofness and to explore by how much exactly we have to weaken ex-post efficiency to arrive at a possibility result.

Most of our results focus on the preference domain where all objects are acceptable. If agents may rank objects as unacceptable and possibly receive no object, notions of

[3] Bogomolnaia and Heo [2012] and Hashimoto et al. [2014] provide axiomatic characterizations of the PS mechanism. Chang and Chun [2017] identify a restricted domain where the impossibility with equal-treatment-of-equals remains unchanged whereas Liu and Zeng [2019] characterize the restricted tier domains where a possibility with equal-treatment-of-equals is obtained (together with strategy-proofness and ex-ante efficiency).

[4] Zhang [2019] proves a strong group-manipulability result, imposing ex-post efficiency and auxiliary fairness axioms that are by themselves weaker than envy-freeness.

[5] Note that in general strategy-proofness and envy-freeness are not equivalent as any serial dictatorship mechanism is strategy-proof but not envy-free, and the mechanism choosing the uniform assignment except for unanimous profiles where the unanimously most preferred assignment is chosen probability one, satisfies envy-freeness but violates strategy-proofness.

efficiency have to take into account the set of (un)assigned objects:[6] a deterministic assignment is (weakly) non-wasteful if no (unassigned) agent prefers an unassigned object to his assignment. As a stronger requirement, ex-ante non-wastefulness demands that if an agent finds an object acceptable but receives no object with positive probability, then the acceptable object must be assigned with probability one. Moreover, if he prefers an object over another and is assigned the less-preferred with positive probability, then the preferred object must be assigned with probability one. Martini [2016] shows that there is no mechanism satisfying strategy-proofness, equal-treatment-of-equals and ex-ante non-wastefulness, i.e., another principal impossibility result on the full domain. In comparison, our impossibility result invokes a stronger notion of fairness and a weaker notion of non-wastefulness. Erdil [2014] studies the ex-ante waste of strategy-proof mechanisms, and shows that RSD is dominated by a strategy-proof mechanism which is less wasteful. It is an open problem to establish the minimal waste in the class of mechanisms satisfying strategy-proofness and equal-treatment-of-equals.

The paper is organized as follows. Section 2 introduces random assignments, their properties and several popular mechanisms. Section 3 contains the impossibility pertaining to unanimity and establishes the upper bound of $\frac{2}{n}$ for q-unanimity among n agents. Section 4 introduces RDcED to show that the bound is tight and characterizes it for three agents. Section 5 presents a novel impossibility result for non-wasteful assignments on the full preference domain and shows how non-wastefulness or unanimity can be weakened to allow for a possibility result. Finally, Section 6 analyses how the upper bound on q-unanimity may be increased by allowing for waste and Section 7 concludes.

2. Model

Let $N = \{1, \ldots, n\}$ denote the set of agents and $O = \{o_1, \ldots, o_n\}$ denote the finite set of objects. Throughout we suppose $|N| = |O| \geq 3$. Each agent i has strict preferences over $O \cup \{i\}$ where i stands for being unassigned; let R_i denote the corresponding linear order[7] and write P_i for its asymmetric part (where xP_iy is defined by xR_iy and $x \neq y$). Let $\mathcal{R}^i$ denote the set of all strict preferences of agent i over $O \cup \{i\}$. Let $\mathcal{R}^N = \times_{i \in N} \mathcal{R}^i$ denote the set of all preference profiles $R = (R_1, \ldots, R_n)$. Let $\underline{\mathcal{R}}^i$ denote the set of all strict preferences of agent i over $O \cup \{i\}$ such that oR_ii for all $o \in O$, i.e., where all objects are acceptable. We denote this domain by $\underline{\mathcal{R}}^N = \times_{i \in N} \underline{\mathcal{R}}^i$ and refer to is as the no-disposal domain, as no agent would ever dispose of any assigned object. For the full domain we write $\mathcal{R}^N$.

[6] Bogomolnaia and Moulin [2015] show PS to achieve the maximal size guarantee among all envy-free mechanisms.

[7] Thus R_i is (i) complete, (ii) transitive and (iii) antisymmetric (xR_iy and yR_ix implies $x = y$).

An assignment is a mapping $\mu : N \to O \cup N$ such that[8] $\mu_i \in O \cup \{i\}$ for all $i \in N$ and $\mu_i \neq \mu_j$ for all $i \neq j$. Let $\mathcal{M}$ denote the set of all assignments.

An assignment μ is efficient under R if there exists no $\mu' \in \mathcal{M}$ such that $\mu'_i R_i \mu_i$ for all $i \in N$ and $\mu'_j P_j \mu_j$ for some $j \in N$. Let $\mathcal{PO}(R)$ denote the set of all efficient assignments under R.

An assignment μ is weakly efficient under R if there exists no $\mu' \in \mathcal{M}$ such that $\mu'_i P_i \mu_i$ for all $i \in N$. Let $\mathcal{WPO}(R)$ denote the set of all weakly efficient assignments under R.

An assignment μ is non-wasteful under R if for all $i \in N$ and all $x \in O \cup \{i\}$, $x R_i \mu_i$ implies there exists $j \in N$ with $\mu_j = x$. Note that this implies $\mu_i R_i i$. Let $\mathcal{NW}(R)$ denote the set of all non-wasteful assignments under R.

An assignment μ is weakly non-wasteful under R if for all $i \in N$ and all $x \in O \cup \{i\}$, $x R_i \mu_i$ and $i R_i \mu_i$ together imply that there exists $j \in N$ with $\mu_j = x$. Again this implies $\mu_i R_i i$. Here it is considered waste if an object is assigned to no one but desired by an *unassigned* agent or if an agent is assigned an unacceptable object. Let $\mathcal{WNW}(R)$ denote the set of all weakly non-wasteful assignments under R. Note that verifying weak non-wastefulness only requires the knowledge of each agent's acceptable objects but no knowledge of the ranking among them (which is necessary to determine non-wastefulness or efficiency of a deterministic allocation).

For any profile R, we have $\mathcal{PO}(R) \subseteq \mathcal{NW}(R) \subseteq \mathcal{WNW}(R)$, and there is no relation between (weak) non-wastefulness and weak efficiency.

Let $\Delta(\mathcal{M})$ denote the set of all probability distributions over $\mathcal{M}$. Given $p \in \Delta(\mathcal{M})$, let p_{ia} denote the associated probability of i being assigned a. Let $supp(p)$ denote the support of p. Then (i) p is ex-post efficient under R if $supp(p) \subseteq \mathcal{PO}(R)$, (ii) p is ex-post weakly efficient under R if $supp(p) \subseteq \mathcal{WPO}(R)$, (iii) p is ex-post non-wasteful under R if $supp(p) \subseteq \mathcal{NW}(R)$, and (iv) p is ex-post weakly non-wasteful under R if $supp(p) \subseteq \mathcal{WNW}(R)$.

For all $i \in N$, all $R_i \in \mathcal{R}^i$ and all $x \in O \cup \{i\}$, let $B(x, R_i) = \{y \in O \cup \{i\} : y R_i x\}$. Then given any $p, q \in \Delta(\mathcal{M})$, p_i stochastically R_i-dominates q_i if for all $x \in O \cup \{i\}$,

$$\sum_{y \in B(x,R_i)} p_{iy} \geq \sum_{y \in B(x,R_i)} q_{iy}.$$

A random assignment p stochastically R-dominate another random assignment q if p_i R_i-dominates q_i for all $i \in N$. A random assignment is sd-efficient if there is no random assignment $q \neq p$ that stochastically R-dominates it.[9] Given two random assignments p and q, we say that p and q are *equivalent* if $p_i = q_i$ for all $i \in N$.

[8]We will use throughout the convention to write μ_i instead of $\mu(i)$ for any $i \in N$.

[9]Bogomolnaia and Moulin [2001] refer to this as 'ordinal efficiency'. It implies Pareto-efficiency with respect to expected utilities for some von Neumann-Morgenstern-representations of agents' ordinal preferences over objects [McLennan, 2002].

A mechanism is a mapping $\varphi : \mathcal{R}^N \to \Delta(\mathcal{M})$. Then $\varphi(R)$ denotes the random assignment chosen for R, and $\varphi_{ia}(R)$ denotes the probability of agent i being assigned object a. Then φ is *sd-efficient* if for all $R \in \mathcal{R}^N$, $\varphi(R)$ is sd-efficient under R. Similarly, we define ex-post (weak) efficiency and ex-post (weak) non-wastefulness for a mechanism.

Then φ is *strategy-proof* if for all $R \in \mathcal{R}^N$, all $i \in N$ and all $R'_i \in \mathcal{R}^i$, $\varphi_i(R)$ stochastically R_i-dominates $\varphi_i(R'_i, R_{-i})$. Note that for any ordinal mechanism (where an agent only submits his ordinal ranking), strategy-proofness is equivalent to the requirement that for any von Neumann-Morgenstern utility presentation of his true ordinal ranking, submitting the true ordinal ranking maximizes his expected utility. Most real-life mechansims only elicit this ordinal information (instead of von Neumann-Morgenstern utilities).

Furthermore, φ is *envy-free* if for all $R \in \mathcal{R}^N$ and all $i \in N$, $\varphi_i(R)$ stochastically R_i-dominates $\varphi_j(R)$ (where in $\varphi_j(R)$ the outside option j is replaced by i). If $\varphi(R)$ attaches probability one to assignment μ, then this is equivalent to $\mu_i R_i \mu_j$ for all $i, j \in N$. Finally, φ is *symmetric* (respectively, treats equals equally) if for all $R \in \mathcal{R}^N$ and all $i, j \in N$, $R_i = R_j$ implies $\varphi_{io}(R) = \varphi_{jo}(R)$ for all $o \in O$.

We also define two invariance conditions of a mechanism with respect to renaming agents and with respect to renaming objects.

Given a permutation $\tau : N \to N$ and $R \in \mathcal{R}^N$, let $\tau(R)$ be the profile such that for all $i \in N$, $\tau(R)_i = R_{\tau(i)}$. A mechanism φ is *anonymous* if for any permutation $\tau : N \to N$ and $R \in \mathcal{R}^N$, we have $\varphi_i(\tau(R)) = \varphi_{\tau(i)}(R)$ for all $i \in N$.

Given a permutation $\sigma : O \to O$ and $R \in \mathcal{R}^N$, let R_i^σ be such that (i) for all $a, b \in O$, aR_ib iff $\sigma(a)R_i^\sigma\sigma(b)$ and (ii) for all $a \in O$, aR_ii iff $\sigma(a)R_ii$, and $R^\sigma = (R_i^\sigma)_{i\in N}$. A mechanism φ is *neutral* if for any permutation $\sigma : O \to O$ and $R \in \mathcal{R}^N$, we have $\varphi_{io}(R) = \varphi_{i\sigma(o)}(R^\sigma)$ for all $i \in N$ and all $o \in O$.

Note that most properties are defined in terms of an agent's random assignment. For a given set of properties, we say that a mechanism φ is *unique in terms of probability shares*, if for any other mechanism ϕ satisfying this set of properties, $\varphi(R)$ and $\phi(R)$ are equivalent for any profile R.

Below we introduce some of the well-known mechanisms on the no-disposal domain.

The uniform assignment (UA) mechanism[10] randomizes uniformly over all $|N|!$ deterministic non-wasteful assignments (irrespective of agents preferences). Hence for individual object assignment probabilities we have: for all $R \in \underline{\mathcal{R}}^N$, $UA_{io}(R) = \frac{1}{n}$ for all $i \in N$ and $o \in O$.

A strict priority ranking over N is denoted by $\succ$. Let $\mathcal{L}$ denote the set of all strict priority rankings. Given $\succ \in \mathcal{L}$, let $f^\succ$ denote the (deterministic) serial dictatorship mechanism where agents are assigned their most-preferred among all available objects

[10] Chambers [2004] characterizes UA via consistency.

in order of their priority.[11] Then the random serial dictatorship (RSD) mechanism is defined by $RSD(R) = \frac{1}{n!}\sum_{\succ \in \mathcal{L}} f^{\succ}(R)$ for all $R \in \underline{\mathcal{R}}^N$.

We omit the formal definition of the probabilistic serial (PS) mechanism[12] and provide an intuitive formulation instead: each agent starts eating with uniform speed from his most-preferred object; once an object is exhausted, each agent eats with uniform speed from his most-preferred among the remaining objects, and so on until all objects are exhausted. The assignment probabilities of any agent in PS are simply the shares of objects the agent has eaten during this process.[13]

3. Unanimity

Efficiency and fairness are particularly hard to reconcile where agents' preferences are in conflict. In search of a possibility result, let us thus begin our analysis by focussing on profiles were preferences are aligned, such that there is unanimous agreement on a unique most-preferred outcome. In our context, these are precisely those profiles where each agent ranks a different object first; we shall refer to them as unanimous profiles. As a restriction of efficiency to this subset of profiles, *unanimity* demands each agent to receive his most-preferred object.[14] Formally, a mechanism φ satisfies unanimity if for any profile $R \in \underline{\mathcal{R}}^N$ where there exists $\mu \in \mathcal{M}$ such that for all $i \in N$ and all $x \in O \cup \{i\}$ we have $\mu_i R_i x$, then $\varphi_{i\mu_i}(R) = 1$ for all $i \in N$. Hence, $\varphi(R)$ attaches probability one to μ when all agents unanimously prefer this assignment to any other assignment (or equivalently, (assignment) unanimity). Note that there is no distinction between an ex-post and ex-ante notion of unanimity; unanimity is implied by both ex-post efficiency and, a fortiori, ex-ante efficiency. Furthermore, as we are interested in the compatibility of envy-freeness and efficiency, unanimous profiles stand out as they are the only profiles where those two requirements are compatible ex-post.

Unfortunately, we find that even this restriction of efficiency to only a small set of profiles is incompatible with strategy-proofness and envy-freeness.

Theorem 1. *On the domain $\underline{\mathcal{R}}^N$ for $|N| \geq 3$, there exists no mechanism which is unanimous, strategy-proof and envy-free.*

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[11]For any $R \in \underline{\mathcal{R}}^N$ and $i_1 \succ i_2 \succ \cdots \succ i_n$, i_1 receives his most R_{i_1}-preferred object in O (denoted by $f^{\succ}_{i_1}(R)$), and for $l = 2, \ldots, n$, i_l receives his most R_{i_l}-preferred object in $O \backslash \{f^{\succ}_{i_l}(R), \ldots, f^{\succ}_{i_{l-l}}(R)\}$ (denoted by $f^{\succ}_{i_l}(R)$).

[12]For that, we refer the reader to Bogomolnaia and Moulin [2001]; Bogomolnaia [2015] offers an alternative definition of PS, and Katta and Sethuraman [2006] extend PS to the domain where indifferences are allowed.

[13]Note that the PS-mechanism pins down individuals' object assignment probabilities directly, rather than a random assignment per se, i.e., a convex combination of deterministic assignments. Nonetheless, corresponding random assignments exists as any bistochastic matrix $(p_{ia})_{i \in N, a \in O}$ can be decomposed as a convex combination of permutation matrices by the Birkhoff-von Neumann Theorem [Birkhoff, 1946].

[14]Observe that unanimous profiles are also the only profiles where there exists a deterministic envy-free assignments other than the trivial assignment leaving all agents unassigned.

We defer the proof of Theorem 1, as it will turn out to be an almost immediate implication of our second, more general, impossibility result.[15]

For that, instead of requiring full unanimity, one may weaken the requirement and demand that at all unanimous profiles, each agent receives their most-preferred object with probability at least q. Formally, given $q \in [0,1]$, a mechanism φ satisfies q-unanimity if for any profile $R \in \underline{\mathcal{R}}^N$ where there exists $\mu \in \mathcal{M}$ such that for all $i \in N$ and all $x \in O \cup \{i\}$ we have $\mu_i R_i x$, then $\varphi_{i\mu_i}(R) \geq q$ for all $i \in N$.[16] Clearly, q-unanimity is compatible with strategy-proofness and envy-freeness for sufficiently low values of q (trivially for $q = 0$). For example the uniform assignment mechanism satisfies the first two properties and $\frac{1}{|N|}$-unanimity.

Before answering for what maximal bound q, q-unanimity can be attained, let us also consider a related requirement, for profiles where there is unanimous agreement that a certain object should be assigned to a specific agent. In particular, for a profile where all agents consider all objects acceptable and the object most-preferred by one agent i is considered least-preferred by all others, efficiency requires that i receives his most-preferred object.[17] We refer to a mechanism ensuring such assignments as object-unanimous. Formally, given $q \in [0,1]$, a mechanism φ satisfies q-object-unanimity, if for any profile $R \in \underline{\mathcal{R}}^N$ where there exists $i \in N$ and $x \in O$ such that for all $y \in O$ we have (i) xR_iy and (ii) yR_jx for all $j \neq i$, then $\varphi_{ix}(R) \geq q$. Given that preferences are aligned with respect to the most-preferred object of agent i,[18] we may ask, as for q-unanimity, for what maximal bound q, q-object-unanimity is compatible with strategy-proofness and envy-freeness.

Theorem 2. *On the domain $\underline{\mathcal{R}}^N$ for $|N| \geq 3$, if a mechanism φ satisfies ex-post weak non-wastefulness, strategy-proofness and envy-freeness, then it fails to be q-unanimous (q-object-unanimous) for any $q > \frac{2}{|N|}$.*

Proof. Let φ be a mechanism satisfying the properties. Since φ satisfies ex-post weak non-wastefulness, we know that assignment probabilities sum to one, both for each agent and hence also for each object; $\sum_{a \in O} \varphi_{ia}(R) = \sum_{i \in N} \varphi_{ia}(R) = 1$ for all $i \in N$, all $a \in O$ and all $R \in \underline{\mathcal{R}}^N$. Let $|N| = n$.

First, we show that φ is q-object-unanimous with $q \leq \frac{2}{n}$ under the auxiliary assumption that φ is neutral.

For that, consider the following profile R (where j stands for all $n-1$ agents other than 1 and where higher ranked objects are preferred over those ranked lower.)

[15]Note that RSD satisfies strategy-proofness, symmetry, unanimity and object-unanimity (which we define below).

[16]This is weaker than requiring μ to be chosen with probability at least q.

[17]Moreover, then for any deterministic assignment μ other than the trivial assignment leaving all agents unassigned, assigning i his most-preferred object under μ is necessary and sufficient for both (i) agent i not to envy any other agent's object ($\mu_i R_i \mu_j$ for all $j \in N$) and (ii) any agent j not to envy agent i's object ($\mu_j R_j \mu_i$ for all $j \in N$).

[18]Note that any random assignment that assigns i their most-preferred object with probability less than one is stochastically R-dominated by another that assigns it with higher probability.

R_1	R_j
o_1	$o_2 : \frac{1}{n-1} - \varepsilon_2$
o_2	$o_3 : \frac{1}{n-1} - \varepsilon_3$
o_3	$o_4 : \frac{1}{n-1} - \varepsilon_4$
$\vdots$	$\vdots$
o_{n-1}	$o_n : \frac{1}{n-1} - \varepsilon_n$
o_n	o_1

On the right hand side one finds the assignment probabilities for the $n-1$ agents 2 to n and the $n-1$ objects o_2 to o_n – by envy-freeness (EF), they all receive objects with the same (object-specific) probability so that by feasibility $\varepsilon_k \geq 0$ for all $k \in \{2, \dots, n\}$. The remaining probabilities follow as residuals – in particular $\varphi_{jo_1}(R) = \sum_{k=2}^n \varepsilon_k$ and $\varphi_{1o_1}(R) = 1 - (n-1)\sum_{k=2}^n \varepsilon_k$. Hence, toward our claim, we will show that $\sum_{k=2}^n \varepsilon_k \geq \frac{n-2}{n(n-1)}$.

Now for any $k \neq 1$, consider the following four profiles.

R_1^k	R_j^k	$R_1^{k'}$	$R_j^{k'}$	$R_1^{k''}$	$R_j^{k''}$	$R_1^{k'''}$	$R_j^{k'''}$
o_1	o_2	o_2	o_2	o_2	o_2	o_2	o_k
o_2	o_3	o_3	o_3	o_3	o_3	o_3	o_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
o_{k-1}	o_k	o_1	o_k	o_1	o_k	o_1	o_{k-1}
o_k	o_1	o_k	o_1	o_{k+1}	o_1	o_{k+1}	o_1
o_{k+1}	o_{k+1}	o_{k+1}	o_{k+1}	$\vdots$	o_{k+1}	$\vdots$	o_{k+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	o_n	$\vdots$	o_n	$\vdots$
o_n	o_n	o_n	o_n	o_k	o_n	o_k	o_n

In R^k, compared to R, agents $j \geq 2$ move o_1 up, just below o_k (for $k = n$ this step is superfluous). By strategy-proofness (SP) and EF this does not change their assignment probabilities for o_k, $\varphi_{jo_k}(R^k) = \frac{1}{n-1} - \varepsilon_k$. Moreover the residual $\varphi_{1o_k}(R^k)$ remains unchanged. Next, in moving to $R^{k'}$, agent 1 demotes object o_1 to just above o_k (for $k = 2$, this step is superfluous). By SP this leaves his, and hence by EF also all j's, probability of receiving o_k unchanged. Moreover at $R^{k'}$, all agents agree on the ranking of objects other than o_1 and o_k and hence receive them with probability $\frac{1}{n}$ by EF. This pins down the assignment probabilities of o_1 as residuals, in particular $\varphi_{jo_1}(R^{k'}) = \frac{n-2}{n(n-1)} + \varepsilon_k$. Again by SP (and EF), the assignment probabilities for object o_1 remain unchanged as we move to $R^{k''}$ and then to $R^{k'''}$ (the first step is superfluous for $k = n$, the second for $k = 2$).

Now, consider the following profile

$\tilde{R}_1$	$\tilde{R}_j$
o_2	o_1
o_3	o_2
o_4	o_3
$\vdots$	$\vdots$
o_n	o_{n-1}
o_1	o_n

By neutrality, we find that for $j \neq 1$ and $k \neq 1$, the probabilities with which j receives object o_k are given by the probabilities with which they receive object o_1 in $R^{k'''}$:

$$\varphi_{jo_k}(\tilde{R}) = \varphi_{jo_1}(R^{k'''}).$$

Hence, the probabilities with which they receive objects o_k, $k \neq 1$, in $\tilde{R}$ sum to $\frac{n-2}{n} + \sum_{k=2}^{n} \varepsilon_k$. As a residual this yields $\varphi_{jo_1}(\tilde{R}) = \frac{2}{n} - \sum_{k=2}^{n} \varepsilon_k$. Since all $n-1$ agents $j \neq 1$ cannot in total receive more than 1 of object o_1, we have

$$1 \geq (n-1)\left(\frac{2}{n} - \sum_{k=2}^{n} \varepsilon_k\right) = 2 - \frac{2}{n} - (n-1)\sum_{k=2}^{n} \varepsilon_k \Leftrightarrow (n-1)\sum_{k=2}^{n} \varepsilon_k \geq \frac{n-2}{n}.$$

Thus, agent 1 may not receive o_1 with probability larger than $\frac{2}{n}$ in profile R. Hence, $\frac{2}{n}$ serves as an upper bound on q-object-unanimity.

By strategy-proofness and envy-freeness, this remains the case as 1 changes the order in which he ranks objects below o_1 or the other agents change the order in which they rank objects above (now they may also differ in their rankings) which completes the proof of our claim. In particular, agents $j \neq 1$ may all rank a different object first. Then 1 is still assigned o_1 with probability at most $\frac{2}{n}$, which serves as an upper bound on q-unanimity.

Finally, assume there exists a non-neutral mechanism which is strategy-proof, envy-free and q-unanimous or q-object-unanimous with $q > \frac{2}{n}$. Then any mechanism derived from it by permuting objects would likewise satisfy these properties – so a uniform mixture over all these permuted mechanisms would restore neutrality and satisfy q-unanimity (q-object-unanimity) with $q > \frac{2}{n}$, a contradiction to the first part. □

As mentioned earlier, Theorem 2 allows us to readily prove Theorem 1. For this, we need the following lemma.

Lemma 1. *On the domain $\underline{\mathcal{R}}^N$, if a mechanism φ satisfies strategy-proofness and envy-freeness, then every agent receives the same total assignment probabilities of objects at any preference profile: for all $R, R' \in \underline{\mathcal{R}}^N$ and all $i, j \in N$,*

$$\sum_{x \in O} \varphi_{ix}(R) = \sum_{x \in O} \varphi_{jx}(R') \equiv Q_\varphi.$$

Proof. For all $i, j \in N$ and all $R, R' \in \underline{\mathcal{R}}^N$ such that $R' = (R'_j, R_{-j})$ we have

$$\sum_{x \in O} \varphi_{ix}(R) = \sum_{x \in O} \varphi_{jx}(R) = \sum_{x \in O} \varphi_{jx}(R'_j, R_{-j}) = \sum_{x \in O} \varphi_{ix}(R'), \tag{1}$$

where the first and last equality follow by envy-freeness while the second equality is due to strategy-proofness. Since any $R^* \in \underline{\mathcal{R}}^N$ can be derived from R by at most n changes in agents' preferences, we have

$$\sum_{x \in O} \varphi_{ix}(R) = \sum_{x \in O} \varphi_{ix}(R^*) = \sum_{x \in O} \varphi_{jx}(R^*) \equiv Q_\varphi,$$

where the first equality follows by iterated application of (1), while the second equality is due to envy-freeness. □

Proof of Theorem 1. By Theorem 2 it remains to show that any unanimous, strategy-proof and envy-free mechanism φ satisfies ex-post weak non-wastefulness. By Lemma 1, φ yields the same total assignment probability Q_φ for each agent and at each profile, including unanimous profiles. As φ is unanimous, we have $Q_\varphi = 1$. Hence, φ is ex-post weakly non-wasteful. By Theorem 2, it can be q-unanimous only for $q \leq \frac{2}{|N|}$ – a contradiction to unanimity. □

Note that Lemma 1 and the above argument also show Theorem 1 to remain unchanged when we replace unanimity by object-unanimity.

Remark 1. *Alternatively one may consider (ex-post) q-efficiency meaning that for any profile the mechanism attaches at least probability q to efficient assignments. Formally, a mechanism φ satisfies q-efficiency if for any profile $R \in \underline{\mathcal{R}}^N$, $\sum_{\mu \in \mathcal{PO}(R)} \varphi(R)(\mu) \geq q$. Now as for deterministic assignments efficiency implies both unanimity and object-unanimity, q-efficiency implies both q-unanimity and q-object-unanimity. Hence, Theorem 2 remains unchanged when q-unanimity is replaced with q-efficiency. Thus, we maintain the same impossibility if we consider the (ex-post) efficiency frontier among strategy-proof and envy-free mechanisms.*[19]

4. Random-Dictatorship-cum-Equal-Division

Theorem 2 provides only an upper bound for q-unanimity. Below we provide a mechanism that achieves this bound, which implies that this bound is tight.

For that, denote agent i's most-preferred object under R_i by $\bar{o}_{R_i}$ and define the *Random-Dictatorship-cum-Equal-Division mechanism (RDcED)* ϕ as follows. For any $i \in N$ and any $R \in \underline{\mathcal{R}}^N$ consider the set of non-wasteful assignments where i receives

[19] Hence, Theorem 1 and Theorem 2 both strengthen Theorem 1 of Nesterov [2017], who finds ex-post efficiency to be incompatible with strategy-proofness and envy-freeness – one may replace ex-post efficiency either by the weaker requirement of unanimity or by the two requirements of q-efficiency (with $q \in (\frac{2}{n}, 1)$) and ex-post weak non-wastefulness that are both weaker than ex-post efficiency. The same remains true for object-unanimity instead of unanimity.

his most-preferred object:

$$\mathcal{M}^i(R) = \{\mu \in \mathcal{M} | \mu_i = \bar{o}_{R_i} \text{ and } \mu_j \in O \text{ for all } j \in N\}.$$

Hence, $\mathcal{M}^i(R)$ contains all $(n-1)!$ assignments resulting from all possible permutations when assigning objects in $O \setminus \{\bar{o}_{R_i}\}$ among agents $N \setminus \{i\}$. Define ϕ^i by taking $\phi^i(R)$ to be the uniform mixture over all $\mu \in \mathcal{M}^i(R)$. In words, the ϕ^i assigns i his most-preferred object before applying the uniform assignment mechanism to all remaining objects and individuals. Finally, define ϕ as the uniform mixture over all ϕ^i:

$$\phi(R) = \frac{1}{n}\phi^i(R).$$

It selects some agent i uniformly at random, lets him choose with highest priority and then divides the remaining objects equally by applying the uniform assignment mechanism.

When all objects are acceptable, the following observations are immediate:

- ϕ is neutral, anonymous, ex-post weakly non-wasteful, strategy-proof and envy-free.
- $\phi(R)$ stochastically-R-dominates the uniform assignment $UA(R)$ – but is itself stochastically-R-dominated by the random assignment under the random serial dictatorship mechanism $RSD(R)$.
- An object, which is either not ranked first by any agent or ranked first by all agents, is assigned uniformly (at random) to each agent with probability $\frac{1}{n}$. In particular, if all agents' most-preferred objects coincide, then we arrive at the uniform assignment.
- An object, which is ranked first by all agents except for i, is never assigned to i, and it is assigned to each $j \neq i$ with equal probability $\frac{1}{n-1}$.
- An object, which is ranked first only by i, is assigned to i with probability $\frac{1}{n} + \frac{n-1}{n}\frac{1}{n-1} = \frac{2}{n}$.
- In general, for the assignment probabilities under RDcED, we find

$$\phi_{io}(R) = \begin{cases} \frac{1}{n} + \frac{n-1}{n}\frac{n-1-|\{j \neq i | o = \bar{o}_{R_j}\}|}{(n-1)^2}, & \text{if } o = \bar{o}_{R_i} \\ \frac{n-1}{n}\frac{n-1-|\{j \neq i | o = \bar{o}_{R_j}\}|}{(n-1)^2}, & \text{if } o \neq \bar{o}_{R_i} \end{cases}.$$

In particular, we find that the upper bounds formulated in Theorem 2 are tight.

Proposition 1. *On the domain $\underline{\mathcal{R}}^N$ for $|N| \geq 3$, the Random-Dictatorship-cum-Equal-Division mechanism (RDcED) satisfies ex-post weak non-wastefulness, strategy-proofness, envy-freeness, $\frac{2}{|N|}$-unanimity and $\frac{2}{|N|}$-object-unanimity.*

Moreover, since under RDcED always at least one agent receives his most-preferred object, RDcED is ex-post weakly efficient. For three agents the latter property together with ex-post weak non-wastefulness, strategy-proofness and envy-freeness characterize RDcED.

Theorem 3. *On the domain $\underaccent{\tilde}{\mathcal{R}}^N$ for $|N| = 3$, the Random-Dictatorship-cum-Equal-Division mechanism ϕ is the unique mechanism in terms of probability shares satisfying ex-post weak efficiency, ex-post weak non-wastefulness, strategy-proofness and envy-freeness.*

Proof. As shown above, ϕ satisfies the properties of Theorem 3. For the other direction, let $N = \{1,2,3\}$ and $O = \{a,b,c\}$. Let φ denote an arbitrary mechanism that satisfies the properties above. We will show that for any profile R, $\varphi(R)$ and $\phi(R)$ are equivalent, i.e., $\varphi_i(R) = \phi_i(R)$ for all $i \in N$.

1. First, we show that whenever some object, say a, is ranked first twice, φ assigns it to both agents ranking it first with probability $\frac{1}{2}$ (and to the third with probability zero). Without loss of generality, let agent 3 be the agent not ranking a first and that 3 ranks c above b. Now, consider the following profile $\tilde{R}$:

$\tilde{R}_1$	$\tilde{R}_2$	$\tilde{R}_3$
a	a	c
b	b	a
c	c	b

By ex-post weak efficiency, 3 cannot receive a with positive probability – in that case, 1 or 2 (say 1) would receive b and the other (say 2) c. But if instead 1 received a, 2 received b and 3 received c, everyone would be strictly better off. So by ex-post weak efficiency, 3 receives a with probability zero and by envy-freeness both 1 and 2 receive a with probability $\frac{1}{2}$ each. Moreover, by strategy-proofness and envy-freeness, the same assignment probabilities hold independently from the order in which 1 and 2 rank b and c. Finally, even as 3 demotes a and ranks it last, strategy-proofness demands that it receives none of it (while the other two still receive it with probability $\frac{1}{2}$ by ex-post weak non-wastefulness and envy-freeness).

2. Next, consider the case where some object, say a, is ranked first by all agents. As we will see, under φ all objects are assigned uniformly. For the case where all agents have the same preferences, this is immediate. If there is some disagreement on the second and third ranked objects, consider w.l.o.g. profile R below along with two other profiles:

R_1	R_2	R_3
a	a	a
b	b	c
c	c	b

R'_1	R'_2	R'_3
b	b	a
a	a	c
c	c	b

R''_1	R''_2	R''_3
b	b	a
a	a	b
c	c	c

In profile R'', we have $\varphi_{3c}(R'') = \frac{1}{3}$ (by envy-freeness) and $\varphi_{3b}(R'') = 0$ (since b is ranked first twice, see **1.** above). By strategy-proofness we thus have $\varphi_{3c}(R') = \frac{1}{3}$ and $\varphi_{3b}(R') = 0$, and by envy-freeness $\varphi_{1c}(R') = \varphi_{2c}(R') = \frac{1}{3}$. By strategy proofness and envy-freeness, 1 and 2 are still assigned c with probability $\frac{1}{3}$ as we first replace R'_1

by R_1 and then R_2' by R_2. Hence c is assigned uniformly in profile R. Of course the same is true for a (by envy-freeness) and, as a residual, for b.

3. Next, we will show that if two agents rank the same object, say b, first while a third, say 3, ranks a different object, say c, first, then 3 will receive his most-preferred object (c) with probability $\frac{2}{3}$. For that, note that the probability with which 3 receives c is independent of the order in which he ranks a and b (by strategy-proofness). Assume he ranks b second, i.e., consider the following profile $\hat{R}$:

$\hat{R}_1$	$\hat{R}_2$	$\hat{R}_3$
b	b	c
$\vdots$	$\vdots$	b
		a

Then we know by the above (**1.**) that $\varphi_{3b}(\hat{R}) = 0$. Moreover for $\check{R}_3 : b\check{R}_3 c\check{R}_3 a$ we know that $\varphi_3(\check{R}_3, \hat{R}_{-3})$ is the uniform lottery (**2.**), so by strategy-proofness, $\varphi_{3c}(\hat{R}) + \varphi_{3b}(\hat{R}) = \frac{2}{3}$. Hence, as claimed, $\varphi_{3c}(\hat{R}) = \frac{2}{3}$.

4. Together, **1.**, **3.**, and envy-freeness pin down the assignment for most profiles where two distinct objects are ranked first. For these profiles, two agents, say 1 and 2, rank the same object, say b, first. If they also agree on the ranking of a and c, then by ex-post weak non-wastefulness and envy-freeness, each receives $\frac{1}{6}$ of the object that 3 ranks first, say c.

If they do not agree on the ranking of a and c, there are two possible profiles (up to a relabelling of 1 and 2 and objects a and c). Consider first profile $\tilde{Q}$:

$\tilde{Q}_1$	$\tilde{Q}_2$	$\tilde{Q}_3$
$\frac{1}{2} : b$	$\frac{1}{2} : b$	$\frac{2}{3} : c$
$\frac{1}{6} + \tilde{\epsilon} : c$	$\frac{1}{3} + \tilde{\epsilon} : a$	$0 : b$
$\frac{1}{3} - \tilde{\epsilon} : a$	$\frac{1}{6} - \tilde{\epsilon} : c$	$\frac{1}{3} : a$

To see that in $\tilde{Q}$, $\tilde{\varepsilon} = 0$, consider a switch by 1 in their ranking of b and c – then they (and 3) receive c with probability $\frac{1}{2}$ (by **1.**) and 2 receives b with probability $\frac{2}{3}$ (by **3.**). By envy-freeness, 1 (and 3) receive b with probability $\frac{1}{6}$. As a residual 1 receives a with probability $\frac{1}{3}$ – and does so even before the switch (by strategy-proofness), so that $\tilde{\varepsilon} = 0$.

5. The only type of profile with two distinct first-ranked objects that remains, is represented by profile Q below where relative to $\tilde{Q}$, 3 has changed the ranking of a and b, now ranking last the object that is ranked first by the other two agents. Consider Q alongside the following three profiles Q', Q'', and Q''':

Q_1	Q_2	Q_3	Q'_1	Q'_2	Q'_3	Q''_1	Q''_2	Q''_3	Q'''_1	Q'''_2	Q'''_3
$\frac{1}{2}:b$	$\frac{1}{2}:b$	$\frac{2}{3}:c$	b	a	c	b	a	c	b	a	b
$\frac{1}{6}+\epsilon:c$	$\frac{1}{3}+\epsilon:a$	$\frac{1}{3}:a$	c	b	a	c	b	b	c	b	c
$\frac{1}{3}-\epsilon:a$	$\frac{1}{6}-\epsilon:c$	$0:b$	a	c	b	a	c	a	a	c	a

Note that Q is just one switch away from the 'Condorcet-cycle' profile Q': if 2 switches a and b we are there. Hence, by strategy-proofness, $\varphi_{2c}(Q') = \varphi_{2c}(Q) = \frac{1}{6} - \epsilon$. At Q' the set of weakly efficient assignments is equal to

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ b & a & c \end{pmatrix}, \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}, \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix} \text{ and } \mu_4 = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}.$$

Note that μ_2 is the only weakly efficient assignment at Q' where agent 2 receives c, and also the only weakly efficient assignment where agent 3 receives a. Hence, by ex-post weak efficiency, $\varphi(Q')$ attaches probability $\frac{1}{6} - \epsilon$ to μ_2, and we have $\varphi_{3a}(Q') = \varphi_{2c}(Q') = \frac{1}{6} - \epsilon$. Between Q'' and Q' agent 3 switches a and b so that by strategy-proofness, $\varphi_{3a}(Q'') \leq \varphi_{3a}(Q') = \frac{1}{6} - \epsilon$. Finally, at Q''', $\varphi_{2a} = \frac{2}{3}$ (by **3.**) and hence $\varphi_{3a}(Q''') = \varphi_{1a}(Q''') = \frac{1}{6}$. As 3 swaps b and c between Q'' to Q''', strategy-proofness implies $\varphi_{3a}(Q'') = \varphi_{3a}(Q''') = \frac{1}{6}$ and we arrive at $\frac{1}{6} = \varphi_{3a}(Q'') \leq \frac{1}{6} - \epsilon$ so that $\varepsilon = 0$.

6. Finally, up to relabelling of objects and agents, there are two types of profiles where three distinct objects are ranked first – either there are three distinct second-ranked alternatives, so that we are in a 'Condorcet cycle' profile, or there is an object that is ranked second twice. For a profile of the first type, consider once more Q'. Looking at Q, strategy-proofness yields $\varphi_{2c}(Q') = \varphi_{2c}(Q) = \frac{1}{6}$. Moreover by ex-post weak efficiency, $\varphi_{3a}(Q') = \varphi_{2c}(Q') = \frac{1}{6}$. In the same way, one can show that $\varphi_{2b}(Q') = \varphi_{1a}(Q') = \frac{1}{6}$ and $\varphi_{1c}(Q') = \varphi_{3b}(Q') = \frac{1}{6}$ which leaves each agent with a probability of receiving their top ranked object with probability $\frac{2}{3}$. Analogously, the same can be shown for any 'Condorcet-cycle' profile, i.e., profiles that differ from Q' by relabelling objects or individuals.

We are left with Q''. By strategy-proofness, $\varphi_{3c}(Q'') = \varphi_{3c}(Q') = \frac{2}{3}$. Also $\varphi_{3c}(Q'') + \varphi_{3b}(Q'') = \varphi_{3c}(Q''') + \varphi_{3b}(Q''') = \frac{5}{6}$, and hence, $\varphi_{3b}(Q'') = \frac{1}{6}$. Then as residual $\varphi_{3a}(Q'') = \frac{1}{6}$ and by envy-freeness, $\varphi_{1a}(Q'') = \frac{1}{6}$. Hence, $\varphi_{2a}(Q'') = \frac{2}{3}$. By strategy-proofness, $\varphi_{2c}(Q'') = \varphi_{2c}(\tilde{Q}) = \frac{1}{6}$. All remaining probabilities follow as residual. □

Observe that neither anonymity and nor neutrality are initially assumed but follow as consequences of the axioms in Theorem 3. To check those are independent, observe that if we dropped ex-post weak non-wastefulness, a random dictatorship where only the dictator is assigned his most-preferred object while everyone else remains unassigned would satisfy all the other axioms. PS satisfies all axioms but strategy-proofness, RSD all but envy-freeness and UA all but ex-post weak efficiency.

Remark 2. *Instead of q-unanimity, we may require exact q-unanimity – where for a unanimous profile each agent receives his most-preferred object with (exact) probability q. Formally, given* $q \in [0,1]$*, a mechanism* φ *satisfies exact q-unanimity if for any profile* R *where there exists* $\mu \in \mathcal{M}$ *such that for all* $i \in N$ *and all* $x \in O \cup \{i\}$ *we have* $\mu_i R_i x$*, then* $\varphi_{i\mu_i}(R) = q$ *for all* $i \in N$*.*

Note that 1-unanimity and exact 1-unanimity are equivalent while exact q-unanimity implies q-unanimity. Hence, Theorem 2 remains unchanged when q-unanimity is replaced with exact q-unanimity. Moreover, as RDcED satisfies exact $\frac{2}{|N|}$*-unanimity, the same holds for Proposition 1.*

5. Unacceptable Objects

Until now our results were confined to the no-disposal domain $\underline{\mathcal{R}}^N$, where every agent finds all objects acceptable. For the full domain $\mathcal{R}^N$, where being unassigned is not necessarily ranked at the bottom of an agent's preference, our impossibility result Theorem 1 remains true, as $\mathcal{R}^N$ is a superdomain of $\underline{\mathcal{R}}^N$. Moreover, as we show below, on this domain we may replace unanimity with ex-post weak non-wastefulness without altering the conclusion of Theorem 1.

Theorem 4. *On the domain* $\mathcal{R}^N$ *for* $|N| \geq 3$*, there exists no mechanism which is ex-post weakly non-wasteful, strategy-proof and envy-free.*

Theorem 4 is an immediate implication of Theorem 1 and the following lemma.

Lemma 2. *On the domain* $\mathcal{R}^N$*, ex-post weak non-wastefulness and strategy-proofness together imply unanimity.*

Proof. Let φ satisfy ex-post weak non-wastefulness and strategy-proofness. Suppose that φ violates unanimity. Then for some profile R and $\mu \in \mathcal{M}$, we have $\mu_i R_i \mu'_i$ for all $i \in N$ and all $\mu' \in \mathcal{M}$ and yet φ does not attach probability one to μ. By ex-post weak non-wastefulness, at least one agent must find at least one object as acceptable under R (as otherwise $\varphi(R)$ attaches probability one to μ by ex-post weak non-wastefulness). Let $\tilde{N}$ denote the set of agents who find at least one object acceptable. For any $i \in N \backslash \tilde{N}$ who finds no object acceptable, ex-post weak non-wastefulness implies $\varphi_{ii}(R) = 1$. So for there to be a violation of unanimity, there exists $j \in \tilde{N}$ such that $R_j : \mu_j \, R_j \, \ldots$ with $\mu_j \in O$ and $\varphi_{j\mu_j}(R) < 1$. Let $R'_j : \mu_j \, R_j \, j \, \ldots$ and $R' = (R'_j, R_{-j})$. By strategy-proofness, $\varphi_{j\mu_j}(R') = \varphi_{j\mu_j}(R) < 1$. Thus, by ex-post weak non-wastefulness, there exists $k \in \tilde{N} \backslash \{j\}$ with $\varphi_{k\mu_j}(R') > 0$. Hence, by ex-post weak non-wastefulness, $R_k : \mu_k \, R_k \ldots \mu_j \, R_k \ldots k$ and $\varphi_{k\mu_k}(R') < 1$. Then k in R' is in the role of j in R and we move to R'' where k is ranked second by k and k receives his most-preferred object μ_k with probability less than one. Again this requires another agent $l \in \tilde{N} \backslash \{j,k\}$ receiving μ_k with positive probability – continuing in this way, we arrive at a profile where all agents in $\tilde{N}$ rank a different object first, consider exactly one object acceptable and yet there is at least one agent in $\tilde{N}$ who remains unassigned with positive probability – a violation of ex-post weak non-wastefulness. □

The above Theorem 4 is the first impossibility result on the full domain not involving any efficiency requirement and only an extremely weak ex-post non-wastefulness notion. It bears resemblance to the impossibility result presented by Martini [2016], who finds that strategy-proofness and the weaker fairness requirement of symmetry are incompatible with a stronger notion of (ex-ante) non-wastefulness. Note that if we content ourself both with the weak fairness requirement of symmetry and our weak non-wastefulness requirement, a possibility result emerges – the random serial dictatorship mechanism satisfies both, together with strategy-proofness.

If we instead insist on strategy-proofness and envy-freeness, the following limited notion of unanimity is jointly compatible: a mechanism φ satisfies *weak unanimity* if for all R where there exists $\mu \in \mathcal{M}$ such that for all $i \in N$ and all $x \in O \cup \{i\} \backslash \{\mu_i\}$ we have both $\mu_i R_i \mu_i'$ and $i R_i \mu_i'$, then $\varphi(R)$ attaches probability one to μ. In other words, if every agent finds at most one object acceptable and no two agents find the same object acceptable, then with probability one each agent who finds no object acceptable receives none while all others receive their only acceptable object. Put differently, weak unanimity demands, that at preference profiles where there exists a unique non-wasteful assignment, that assignment is chosen with probability one – compared to unanimity that demands an assignment to be chosen whenever it is the unique efficient assignment. Hence weak unanimity not only weakens unanimity, but is also implied by weak non-wastefulness.

Proposition 2. *On the domain $\mathcal{R}^N$ for $|N| \geq 3$, there exists a mechanism satisfying weak unanimity, strategy-proofness and envy-freeness.*

We show Proposition 2 by extending the uniform assignment mechanism UA to the full domain as follows. For any $R \in \mathcal{R}^N$, define assignment probabilities in two steps:

(1) For any object a, let $n_a(R)$ denote the number of agents who consider a acceptable; then assign each agent i who finds a acceptable a probability share of $s_{ia}(R) = \frac{1}{n_a(R)}$ of the object (i.e., we split the object equally among all agents who consider it acceptable).

(2) For all $i \in N$ and all $a \in O$, choose $UA_i(R)$ such that

$$\sum_{x \in B(a,R_i)} UA_{ix}(R) = \min\left\{ \sum_{x \in B(a,R_i)} s_{ix}(R), 1 \right\}.$$

That is, agent i receives each object a with probability of at most their probability share s_{ia}, starting from his most-preferred object and up until either his probability shares are exhausted or until his total assignment probabilities sum to one.

As with the probabilistic serial, this definition only pins down individual assignment probabilities rather than a convex combination of deterministic assignments $\mu \in \mathcal{M}$. However, the Birkhoff-von Neumann Theorem ensures that such convex combinations exist. Moreover, the individual assignment probabilities suffice to verify that our extension of *UA* satisfies the alleged properties.

One easily verifies that *UA* is strategy-proof – failing to report an acceptable object as acceptable only shrinks the budget set of probability shares while reporting an unacceptable object as acceptable may only worsen the choice from that budget set. Similarly, envy-freeness is readily verified, as two individuals who both consider an object acceptable will receive an equal probability share towards their budget set, while an agent that finds an object unacceptable will not envy others for being assigned that object. Last, to verify weak unanimity, observe that at a profile where each agent finds a different, and only one, object acceptable, each agent will receive a probability share of one of their only acceptable object, and hence is assigned that object for sure.

To see that *UA*, while satisfying the properties in Proposition 2, fails to be ex-post weakly non-wasteful on $\mathcal{R}^N$, as implied by Theorem 4, consider the following example.

Example 1.

R_1	R_2	R_3
a	b	c
b		

Here agent 1 *will receive a probability share of* 1 *of object a and hence, as it is also his most-preferred object, be assigned that object for sure under UA. In terms of probability shares, object b is split between agents* 1 *and* 2*, with each receiving* $\frac{1}{2}$ *but only* 2 *may be assigned the object, with probability* $\frac{1}{2}$*. As b is* 2*'s only acceptable object, he will remain unassigned with equal probability and, in this case, find that b has not been assigned – a violation of ex-post weak non-wastefulness.*

6. Allowing for Waste when All Objects are Acceptable

Theorem 2 shows that for ex-post weakly non-wasteful mechanisms, strategy-proofness and envy-freeness imply that q-unanimity (q-object-unanimity) may only be satisfied for $q \leq \frac{2}{|N|}$ on the no-disposal domain.

What if we were to accept waste (even though all agents find all objects are acceptable), i.e., to allow that the total probability with which any agent receives an acceptable object (Q_φ) was reduced below 1? While envy-freeness and strategy-proofness still ensure that the same Q_φ holds for all *agents* and across all profiles, the total assignment probabilities for *objects* may vary, both within and across profiles when $Q_\varphi < 1$. As the following example demonstrates, this additional flexibility allows to increase the upper bound on q-unanimity that is achievable – mechanism φ below satisfies strategy-proofness, envy-freeness, and q-unanimity for $q = \frac{5}{6} = Q_\phi$. Hence, ex-post weak non-wastefulness is indispensable in Theorem 2.

Example 2. *Let $N = \{1,2,3\}$ and $O = \{a,b,c\}$. For all $R \in \underline{\mathcal{R}}^N$, $\varphi(R)$ is defined as follows (where $Q_\varphi = \frac{5}{6}$ so that any agent is always unassigned with probability $\frac{1}{6}$):*

(1) If under R all agents rank different objects at the top, then each agent receives their top ranked object with probability $\frac{5}{6}$.
(2) If under R all agents rank the same object at the top, then each agent receives their top ranked object with probability $\frac{1}{3}$, their second ranked object with probability $\frac{1}{3}$ and their third ranked object with probability $\frac{1}{6}$.
(3) Otherwise two agents rank the same object at the top, say 1 and 2 rank a at the top, while the remaining agent 3 ranks a different object, say b, at the top; then agent 3 receives his top ranked object with probability $\frac{1}{2}$, his second ranked object with probability $\frac{1}{6}$ and his third ranked object with probability $\frac{1}{6}$; any agent ranking a first and c second receives a and c with probability $\frac{5}{12}$ each; any agent ranking a first and b second a with probability $\frac{5}{12}$, b with probability $\frac{1}{4}$, and their third ranked object c with probability $\frac{1}{6}$.

It is straightforward that φ satisfies $\frac{5}{6}$-unanimity, envy-freeness, anonymity and neutrality. We check that ϕ satisfies satisfies strategy-proofness: if R is in (1), then this is immediate; if R is in (2), then an agent can only deviate to another profile in (2) by reversing the ranking of his second and third most-preferred object (worsening his outcome) or to a profile in (3) (but then he receives the most-preferred object with probability $\frac{1}{6} < \frac{1}{3}$ and the least preferred object with probability at least $\frac{1}{6}$). If R is in (3), then

- *agent 3 can only deviate to another profile in (3) (reverse the ranking of a and c with no change or rank c first with a worse outcome) or to a profile in (2) by ranking a first – but then he gets weakly less of his most-preferred object (at most $\frac{1}{3}$ instead of $\frac{1}{2}$) and of his two most-preferred objects (at most $\frac{2}{3}$ instead of $\frac{2}{3}$);*
- *agent 1 (or 2) who ranks c second can deviate to another profile in (3) by reversing the order of b and c (worsening his outcome) or by ranking b first (but then he gets only $\frac{5}{12}$ of his two most-preferred objects a and c), or to a profile in (1) by ranking c first (but then he gets none of his most-preferred object and still $\frac{5}{6}$ of his two most-preferred objects);*
- *agent 1 (or 2) who ranks b second can deviate to another profile in (3) by reversing the order of b and c (worsening his outcome) or by ranking b first (but then gets $\frac{1}{4}$ or none of his most-preferred object a and still at most $\frac{2}{3}$ of his two most-preferred objects a and b), or to a profile in (1) by ranking c first – but then he gets none of his two most-preferred objects a and b;*

While waste in the above sense is a serious curtailment of efficiency, it raises the question whether a 'small' amount of waste is sufficient to construct an 'almost' unanimous mechanism. To be precise, let us write q^n for the highest possible value q, such that for an economy of size n, i.e., $|N| = |O| = n$, there exists a strategy-proof and envy-free mechanism that is q-unanimous.[20] Do there exist markets of size n such that q^n can be arbitrarily close to 1? Unfortunately our hopes are dashed once again. In fact, as the market grows in size, the upper bound on feasible q-unanimity is ever shrinking.

Theorem 5. *Let q^n be the maximal value q such that there exists a strategy-proof and envy-free mechanism on the domain $\underline{\mathcal{R}}^N$, $|N| = n$, that is q-unanimous. Then q^n is weakly decreasing in n.*

Proof. Consider $N = \{1, \dots, n\}$, $O = \{o_1, \dots, o_n\}$ as well as $N' = N \cup \{n+1\}$ and $O' = O \cup \{o_{n+1}\}$. For any mechanism φ' on $\underline{\mathcal{R}}^{N'}$ that is strategy-proof, envy-free and q-unanimous for some $q \in [0,1]$, we show that there exists another mechanism φ on $\underline{\mathcal{R}}^N$ that satisfies the same properties – in particular one that is at least q-unanimous for the same q. As q^n and q^{n+1} are defined as the maximal feasible bounds on q-unanimity, this implies that the bound q^n must be at least as high as the bound q^{n+1}.

For that, construct φ for each profile $R \in \underline{\mathcal{R}}^N$ as follows: extend preferences R_i by letting each agent $i \in N$ rank object o_{n+1} at the last position to arrive at R'_i, and let agent $n+1 \notin N$ rank o_{n+1} as the most-preferred object. Further, we may fix agent $n+1$'s preferences in any arbitrary order, say as follows (where $i \in N \backslash \{n+1\}$):

R'_i	R'_{n+1}
$\vdots$	o_{n+1}
$\vdots$	o_n
$\vdots$	$\vdots$
o_{n+1}	o_1

Define the set of profiles $R' \in \underline{\mathcal{R}}^{N'}$ thus derived as $\mathcal{Q}^{N'} \subset \underline{\mathcal{R}}^{N'}$ and set $\varphi_{ia}(R) = \varphi'_{ia}(R')$ for any $i \in N$, $a \in O \cup \{i\}$ and all $R \in \underline{\mathcal{R}}^N$. Since φ' is (by assumption) strategy-proof and envy-free on $\underline{\mathcal{R}}^{N'}$, it is a fortiori strategy-proof and envy-free on $\mathcal{Q}^{N'}$.

But then φ is also strategy-proof on $\underline{\mathcal{R}}^N$, as reporting $\tilde{R}_i$ instead of R_i yields a stochastically dominated random assignment over objects in O:

$$\sum_{o_j \in B(o_k, R_i)} \left(\varphi_{io_j}(R) - \varphi_{io_j}(\tilde{R}_i, R_{-i})\right) = \sum_{o_j \in B(o_k, R'_i)} \left(\varphi'_{io_j}(R') - \varphi'_{io_j}(\tilde{R}'_i, R'_{-i})\right) \geq 0$$

[20]As for any economy of size n, and for any labelling of objects, O, and agents, N, the set of mechanisms is closed and bounded in $\mathbb{R}^s$ (where $s = \left(|O|!^{|N|}\right) \times \left(|N| \times |O|\right) = n!^n \times n^2$), this maximum is well-defined for any given N and O. As the maximum does not depend on the labelling of objects and agents, q^n is well-defined for any given n.

for all $o_k \in O$ and all $i \in N$, where the inequality follows from the fact that φ' is strategy-proof on $\mathcal{Q}^{N'}$. In the same way, envy-freeness of φ' on $\mathcal{Q}^{N'}$ implies envy-freeness of φ on $\underline{\mathcal{R}}^N$. To see that φ is q-unanimous, consider any profile $R \in \underline{\mathcal{R}}^N$ where all $i \in N$ rank a different object $o_k \in O$ first. Then at the associated profile R' each agent $i \in N \cup \{n+1\}$ ranks a different object $o_k \in O \cup \{o_{n+1}\}$ first. Hence, by our assumption on φ', each agent receives their most-preferred object with probability at least q at $\varphi'(R')$ and by our construction of φ the same holds for all $i \in N$ at $\varphi(R)$. □

It follows from Theorem 2 that $q^3 < 1$ (as otherwise $q^3 = 1$ and by Lemma 1 there would exist a mechanism satisfying ex-post weak non-wastefulness, strategy-proofness and envy-freeness). This together with Theorem 5 yields that for any $|N| \geq 3$, the maximal bound q, such that there exist a strategy-proof and envy-free mechanism which is q-unanimous, is weakly lower than $q^3 < 1$, i.e., q is uniformly bounded away from one for all n. A fortiori, the same holds for mechanisms which are in addition q-object-unanimous. As a final result, we determine the maximal bound for such mechanisms for three agents.

Theorem 6. *On the domain $\underline{\mathcal{R}}^N$ with $N = \{1, 2, 3\}$ and allowing waste, there exists a strategy-proof and envy-free mechanism satisfying q-unanimity and q-object-unanimity if and only if $q \leq \frac{17}{18}$.*

Proof. Note that $\frac{17}{18} = \frac{85}{90}$ and we specify the probabilities in terms of shares of $\frac{1}{90}$. Consider the following nine profiles.

R_1^1	R_2^1	R_3^1
a^{85}	b^{85}	c^{85}
$\vdots$	$\vdots$	$\vdots$

R_1^2	R_2^2	R_3^2
a^{40}	a^{40}	c^{85}
b^{45}	b^{45}	a^{0}
c^{0}	c^{0}	b^{0}

R_1^3	R_2^3	R_3^3
a^{40}	a^{40}	c_x^{60}
b^{25}	c^{25}	$a_{67.5-x}^{7.5}$
c^{0}	$b^{17.5}$	$b^{17.5}$

R_1^4	R_2^4	R_3^4
a^{40}	a^{40}	c^{45}
c^{15}	c^{15}	a^{10}
b^{30}	b^{30}	b^{30}

R_1^5	R_2^5	R_3^5
a^{45}	a^{45}	c^{85}
b^{40}	b^{40}	b^{0}
c^{0}	c^{0}	a^{0}

R_1^6	R_2^6	R_3^6
a^{45}	a^{45}	c_x^{60}
b^{40}	c_y^{15}	b_{85-x}^{25}
c^{0}	b_{40-y}^{25}	a^{30}

R_1^7	R_2^7	R_3^7
a^{45}	a^{45}	c^{45}
c^{40}	c^{40}	b^{40}
$b^{17.5}$	$b^{17.5}$	a^{0}

R_1^8	R_2^8	R_3^8
a^{30}	a^{30}	a^{30}
b^{25}	b^{25}	b^{25}
c^{30}	c^{30}	c^{30}

R_1^9	R_2^9	R_3^9
a^{30}	a^{30}	a^{30}
$b^{37.5}$	$b^{37.5}$	c^{55}
$c^{17.5}$	$c^{17.5}$	b^{0}

First, it can be verified that the neutral mechanism ϕ defined by the above nine profiles is strategy-proof and envy-free, and satisfies both $\frac{17}{18}$-unanimity and $\frac{17}{18}$-object-unanimity. Note that by neutrality, for any profile $R \in \underline{\mathcal{R}}^N$, we have one of the following cases: (i) all agents rank a different object first (which corresponds to R^1), (ii) all agents rank the same object first (which corresponds to R^8 or R^9), (iii) two agents rank one object first and they rank the third agent's top object last (which corresponds to R^2 and R^5), (iv) two agents rank one object first and they rank the third agent's top object second (which corresponds to R^4 and R^7), or (v) two agents rank one object first, one of them ranks the third agent's top object last and the other one second (which corresponds to R^3 and R^6). Thus, as ϕ is neutral, ϕ is completely defined by the nine profiles above.

Second, let φ be a strategy-proof and envy-free mechanism satisfying q-unanimity and q-object-unanimity. Then, without loss of generality, we may suppose that φ is neutral (by the argument in the proof of Theorem 2) and that φ always assigns probability q to any agent to all real objects (as otherwise we subtract $Q_\varphi - q$ for any $\varphi_i(R)$ from the lowest ranked objects, and the resulting mechanism continues to satisfy all of the properties). Suppose that $q > 85$ (where again we write 85 instead of $\frac{85}{90}$). We show that φ must choose the same probabilities as above except for where we have specified lower indices (in profiles R^3 and R^6).

Obviously, by q-object-unanimity, $\varphi_{3c}(R^5) = q > 85$. Furthermore, by q-unanimity, $\varphi_{1a}(R^1) = \varphi_{2b}(R^1) = \varphi_{3c}(R^1) = q$.

Consider profile R^9: we have by feasibility, $\varphi_{3a}(R^9) \leq 30$, and by strategy-proofness (from R^5), $\varphi_{3c}(R^9) > 55$ and $\varphi_{3b}(R^9) = 0$. Thus, by envy-freeness and feasibility, $\varphi_{1c}(R^9) = \varphi_{2c}(R^9) < 17.5$

Now we have $\varphi_{1a}(R^5) = \varphi_{2a}(R^5) \leq 45$ implying (by using R^1 and strategy-proofness) $\varphi_{1b}(R^5) = \varphi_{2b}(R^5) > 40$.

But now we have in R^7, $\varphi_{3a}(R^7) = 0$ (as agent 3 can move b to the top and receive by q-object-unanimity probability q for object b). By invoking strategy-proofness for 1 and 2 from R^9 (by switching the ranking of a and b), and then using neutrality to arrive at R^7, we have $\varphi_{1b}(R^7) = \varphi_{2b}(R^7) < 17.5$. Thus, $\varphi_{1c}(R^7) = \varphi_{2c}(R^7) > 22.5$ (as $\varphi_{1a}(R^7) = \varphi_{2a}(R^7) \leq 45$). Thus, $\varphi_{3c}(R^7) < 45$.

But now as agent 3 moves from R^7 to R^4 by switching the ranking of a and b, we have $\varphi_{3c}(R^4) < 45$. In R^2, we have $\varphi_{1a}(R^2) = \varphi_{2a}(R^2) > 40$ (from R^1). Hence, $\varphi_{1a}(R^4) = \varphi_{2a}(R^4) > 40$, and by feasibility, $\varphi_{3a}(R^4) < 10$. But then by envy-freeness, $\varphi_{3b}(R^4) \leq 30$. Hence,

$$q = \varphi_{3a}(R^4) + \varphi_{3b}(R^4) + \varphi_{3c}(R^4) < 10 + 30 + 45 = 85,$$

which is a contradiction to $q > 85$. Hence, we must have $q = 85$.

Now all the probabilities can be derived in straightforward manner using the above arguments if $q = 85$. □

It is worth mentioning that for any mechanism achieving the maximal bound in the proof of Theorem 6, that for the profile R^8 where all agents have identical preferences, the middle ranked object is wasted whereas the top ranked and last ranked object are assigned with probability one.

7. Conclusion

We have determined the possibility frontier in terms of unanimity for assignment mechanisms that are both strategy-proof and envy-free. Even for preference profiles where there is unanimous agreement on the preferred assignment, it can only be chosen with a small probability on which we provide an exact bound.

Moreover, for our new mechanism, RDcED, which lies on the possibility frontier in terms of unanimity, is characterized by a natural set of axioms for three agents, we find that while it improves upon RSD in that it is envy-free, it is also Pareto-dominated by the latter. That is, while RSD may yield random assignments where an agent may envy another, RDcED avoids this, but only by making both agents worse off – a rather unsatisfactory solution.[21]

Our results narrow the limits for efficiency under the considered constraints. Below we describe alternative avenues for future research.

On the one hand, in small markets, one could explore which gains in efficiency are achievable by contenting ourselves with weaker notions of incentive compatibility than strategy-proofness or weaker notions of fairness than envy-freeness. While such weakenings can be formulated in the current framework where agents compare random assignments based on stochastic dominance,[22] another approach would be to base agents' comparisons of random assignments in terms of their associated expected utility – while the notion of (ex-ante) efficiency would become stronger by such a change, strategy-proofness and envy-freeness become easier to satisfy, potentially creating room for Pareto-improvements relative to RDcED. Note, however, that eliciting information on agents' von Neumann-Morgenstern utility functions, underlying their ordinal ranking of objects, may be difficult in practice. Here additional research on real-life elicitation procedures might be useful.

On the other hand, one may consider large markets. Here one may make markets large in two different ways: either by keeping the set of object types fixed and adding copies to match an increasing number of agents, or by considering economies with a large number of distinct agents and distinct objects. First, when we add object copies, Liu and Pycia [2016] have shown in their Theorem 2 that any two symmetric and "regular"[23] mechanisms, which are asymptotically strategy-proof and asymptotically efficient, coincide asymptotically, i.e., they choose the same allocations

[21] Note how this contrasts with the relation between two strategy-proof mechanisms in a related setting were agents enjoy different, object-specific priorities: Deferred Acceptance (DA) and Top Trading Cycles (TTC). Here, the first avoids justified envy while the latter is efficient – however, TTC does not Pareto-dominate DA.

[22] See for example, Basteck [2018] for an extensive analysis of logical relationships between various fairness concepts in the present context.

[23] Loosely speaking, this means that agents cannot change to "too much" the random assignments of other agents (in terms of probability shares) as the market becomes large.

in the limit. For instance, this implies asymptotic coincidence of RSD[24] and PS (which was first shown by Che and Kojima [2010]), and that RSD and, respectively, PS satisfy unanimity, object-unanimity and asymptotically both strategy-proofness and envy-freeness. An open question is, whether there exists a sequence of mechanisms, converging to RSD and PS as we add object copies, which is strategy-proof and envy-free for all finite markets. Second, when we consider economies with a large number of distinct agents and distinct objects, Manea [2009] has shown that RSD is sd-efficient with probability zero, and hence RSD and PS diverge with probability one. As we have shown, the extent to which q-(object-)unanimity can be attained in the class of strategy-proof and envy-free mechanisms is bounded away from one when we increase both the number of agents and objects n, but it is an open question whether the bound of $q = \frac{2}{n}$ can be substantially improved even for larger markets. Hence, we do not obtain any extreme convergence result à la Liu and Pycia [2016] or any extreme divergence result à la Manea [2009].

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