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Discussion Paper

SP II 2019–201

March 2019

WZB Berlin Social Science Center

Research Area
Markets and Choice

Research Unit
Market Behavior

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Abstract

Hiring from a pool of workers

by Azar Abizada and Inácio Bó*

We consider the hiring of public sector workers through legislated rules and exam-based rankings, as is done in many countries and institutions around the world. In them, workers take tests and are ranked based on scores in exams and other pre-determined criteria, and those who satisfy some eligibility criteria are made available for hiring in a "pool of workers." In each of an ex-ante unknown number of rounds, vacancies are announced and workers are then hired from that pool. We show that when the scores are the only criterion for selection, the procedure satisfies desired fairness and independence properties. We show, with the aid of details of procedures used in Brazil, France and Australia, that when compositional objectives are introduced, such as affirmative action policies, both the procedures used in the field and in the literature fail to satisfy those properties. We then present a new rule, which we show to be the unique rule that satisfies those properties. Finally, we show that if multiple institutions hire workers from a single pool, even minor consistency requirements are incompatible with compositional objectives.

Keywords: public organizations, hiring, affirmative action

JEL classification: C78, J45, L38, D73

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The authors thank Oguz Afacan, Samson Alva, Janaína Gonçalves, Rosalia Greco, Rustamdjan Hakimov, Philipp Heller, Sinan Karadayi, Morimitso Kurino, Alexander Nesterov, Thibaud Pierrot, Bertan Turhan, and Chiu Yu Ko for helpful comments. Research assistance by Florian Wiek is greatly appreciated. Financial support from the Leibniz Association through SAW project MADEP is gratefully acknowledged.

1. Introduction

While most companies are free to use almost any criteria to decide which workers to hire and when, that is not the case in many governments and institutions around the world. In order to reduce the agency problems of government institutions and increase the transparency of the hiring process, those institutions have to follow clear and strict criteria for selecting workers. In particular, when the number of workers hired is large (such as police officers, tax agents, etc,) the selection procedure may consist of several steps, such as written exams, physical and psychological tests, interviews, and so on, which may also be time consuming. Due to the high costs of executing such selection procedures, these hirings often take place in two phases: the evaluation phase, in which workers apply for the job and take part in the above-mentioned tests and exams, and the second phase, in which the institutions select, over time and on a need basis, workers from the "pool" of workers who took part in the first phase. After a certain period of time, the pool of workers is renewed, with new ones coming through a new evaluation phase. As described by the Public Service Commission of the New South Wales government:

> "A talent pool is a group of suitable candidates (whether or not existing Public Service employees) who have been assessed against capabilities at certain levels. (...) Using a talent pool enables you to source a candidate without advertising every time a vacancy occurs. You can either directly appoint from the pool without further assessment (for example, to fill a shorter term vacancy), or conduct a capability-based behavioural interview with one or more candidates from the pool to ensure a fit with organizational, team and role requirements (and/or additional assessment for agency, role specific or specialist requirements – this is recommended for longer term or ongoing vacancy). This considerably reduces the time and costs associated with advertising."[1]

The main characteristics of these procedures, which will be essential to our analysis, are that (i) the selection of workers to hire, at any time, follows a well-defined rule, which is a systematic way of selecting workers to fill a specified number of positions, (ii) workers are hired in *rounds*, on a need-basis, and must be selected from the pool of workers who took part in the evaluation phase, and (iii) the institutions do not necessarily know ex-ante the number of workers that they will hire during the period of validity of the pool. Therefore, in general, not all workers in the pool will be hired. This aspect is emphasized in the description of the selection process used for all personnel hiring in the European Union institutions:

[1]Source: Public Service Commission of the New South Wales (http://www.psc.nsw.gov.au/employmentportal/recruitment/recruitment/guide/planning/talent-pools)

> "The list is then sent to the EU institutions, which are responsible for recruiting successful candidates from the list. **Being included on a reserve list does not mean you have any right or guarantee of recruitment.**" [emphasis from original article][9]

Very often, the rules used for hiring workers involve scores in the selection process. This is not uncommon: the criteria that is used mostly consists of a weighted average of performance points in multiple dimensions, such as written exam results, education level, etc.[2] When the workers' scores constitutes the sole element for determining which workers to hire, a very natural rule, namely *sequential priority*, is commonly used: if q workers are to be hired, hire the q workers with the highest scores among those in the pool. This rule is simple but has many desirable characteristics. First, it is fair in the sense that every worker who is not (yet) hired has a lower score than those who were hired. This adds a strong element of transparency to the process: if the worker can see, as is often the case, the scores of those who were hired (or at least the lowest score among those who were hired), then she has a clear understanding of why she was not hired. Secondly, it responds to the agency problem: an institution cannot arbitrarily select low-scoring workers before selecting all the ones who have a score higher than that worker. Finally, the quality and identity of the selected workers does not depend on the number of rounds and vacancies in each round. That is, selecting 20 workers in four rounds with five workers in each results in selecting the same workers as if 20 workers were selected at once. We denote this property by *aggregation independence*. One implication of this requirement is that the set of selected workers is independent of the number of rounds and vacancies in each round: selecting 10 workers in two rounds of five workers in each results in the same selection as selecting two workers in each of five rounds.

While sequential priority satisfies those desirable properties, the criteria used for hiring workers often combine scores with other compositional objectives. In section 4 we evaluate rules that are used in real-life applications in different parts of the world, which combine scores with additional objectives. These include "quotas" for individuals with physical or mental disabilities in public sector jobs in France, for black workers in public sector workers in Brazil, as well as the gender balanced hiring of firefighters in the Australian province of New South Wales. We also consider a natural extension of the use of minority reserves, a procedure introduced in the literature of matching with distributional objectives. We show that all of these fail most of the time to satisfy natural concepts of fairness and aggregation independence. These problems are in part explained in section 5, where we show that a new rule that we propose is the unique rule

[2]Real-life examples of selection rules based on the ranking of workers are the selection of policemen in Berlin and public sector workers in Brazil and France.

that satisfies the concept of fairness adapted to the compositional objectives considered and is aggregation independent.

In section 6 we consider the cases where there are multiple institutions (or locations, departments, etc.) hiring from a single pool of workers. While this scenario is very common, our main result shows that a very mild requirement, saying that the order in which firms hire workers should not change whether a worker is hired by some of the institutions, essentially leaves us with a single rule, which says that all institutions must hire workers following a single common priority over them. As a result, no compositional objectives are possible in these scenarios.

Other than the sections described above, the rest of the paper is organized as follows. In section 2 we introduce the basic model of hiring by rules and justify the desirability of aggregation independence. In section 3 we restrict our focus for rules that are based on scores associated with each worker, and in section 7 we conclude. Proofs and formal descriptions of the rules absent from the main text are found in the appendix.

1.1. **Related literature.** The structure and functioning of the hiring process for public sector workers has many elements that makes it a clear target for market design: salaries and terms of employment are often not negotiable, the criteria for deciding who should be hired are exogenously given (or "designed") and there is a clear concern with issues of fairness and transparency. This paper is, to the extent of our knowledge, the first to evaluate from a theoretical perspective this type of hiring that takes place in the public sector, in which workers are sequentially hired following a pre-determined criterion.

There are a few branches of the literature, however, that are related to our analysis. First, the description and analysis of methods for hiring public sector workers around the world, and the incentives involved. [12] evaluates to what extent the use of examinations constitute a meritocratic method for recruiting in the public sector. The author observes that exams may not be the most adequate way to identify fitness for each function, but that the patronage risk that is involved when using more subjective criteria such as interviews and CV screening often overcomes those losses. In fact, in an empirical analysis in different ministries in the Brazilian federal government, [6] find a positive relation between corruption cases and the proportion of employees hired by using subjective criteria.

The property of aggregation independence, that we propose is important for this problem, is somewhat related to notions of consistency [14, 13, 15]. Loosely speaking, an allocation rule is consistent if whenever agents leave the problem with their own allocations, the solution of the residual problem makes the same allocation among the remaining agents. Aggregation independence, on the other hand, says that the order (or timing) in which the allocation of a given number of jobs take place does not change the identity of those who will get the jobs. Different notions of consistency have been used in other matching and allocation problems based on priorities as well [8, 11].

Finally, a big part of our analysis concerns what we denote compositional objectives, that is, objectives regarding characteristics that some portions of the workers hired should have, such as minimum proportions of workers with disabilities, from ethnic minorities or from certain genders. A number of papers have recently tackled these issues from a market design perspective in school choice [2, 10, 7, 5] and college admissions [1, 4].

2. Hiring by rules and aggregation independence

A **rule** determines which workers an institution should hire, given a number of workers to hire, a pool of workers and, potentially, the workers that the institution hired before. Each time an institution attempts to hire workers from the pool is denoted a **round**.

Let A be the set of workers hired in previous rounds, and W be the pool of workers available. For each (W, A, q), a rule φ determines which q workers from W should be hired. That is, $\varphi(W, A, q) \subseteq W$ and $|\varphi(W, A, q)| = \min\{q, |W|\}$. For simplicity of notation, we will sometimes use the following shorthand:

$$\varphi(W, A, \{q_1, \dots, q_t\}) \equiv \varphi\left(W^0, A^0, q_1\right) \cup \varphi\left(W^1, A^1, q_2\right) \cup \cdots \cup \varphi\left(W^{t-1}, A^{t-1}, q_t\right)$$

Where $A^0 = \emptyset$, $W^0 = W$ and for $i > 0$, $A^i = A^{i-1} \cup \varphi\left(W^{i-1}, A^{i-1}, q_i\right)$ and $W^i = W \backslash A^i$. Also for simplicity, we will use the shorter notation $\varphi(W, q)$ when $A = \emptyset$. Unless stated explicitly, none of our results rely on situations in which there are not enough workers, either in general or with some characteristics, to be hired. That is, in all of our results we will assume that the number of workers in W is at least as large as $\sum q_i$, and the same holds for the cases that we will evaluate in which some workers belong to minority groups.

One crucial property of the process of hiring by rules is that the sequence of hires $(q_1, \dots, q_t)$ is ex-ante unknown. That is, every round may or may not be the last one. The total number of workers who will be hired is also unknown. Therefore, the properties that we will deem as desirable should hold at any point in time. In this context, a critical property that a rule should satisfy is **aggregation independence**. A rule is aggregation independent if the total set of workers hired after a certain number of rounds does not depend on how these hires are distributed among rounds.

Definition 1. A rule φ is **aggregation independent** if for any $q \geq q_1 \geq 0$ and sets of workers W and A, $\varphi(W, A, q) = \varphi(W, A, \{q_1, q - q_1\})$.

Therefore, when the rule being used is aggregation independent, an institution that hires q_1 workers in the first round and q_2 in the second will select the same workers that it would by hiring $q_1 + q_2$ in a single round. In fact, one can easily check that if a rule is aggregation independent, this extends to any combination of rounds: if $\sum_i q_i^a = \sum_j q_j^b$, a sequence of hires $q_1^a, \ldots, q_n^a$ will select the same workers as $q_1^b, \ldots, q_m^b$.

We now provide three reasons to justify aggregation independence as a strongly desired property for rules for hiring by rules: transparency, non-manipulability, and robustness.

Transparency

One of the main reasons driving governments and institutions to use hiring by rules is that, for those who are not hired, the reason why is made clearly and simply. Take, for example, the rule that consists of always hiring the workers with the highest scores in an exam. By knowing the rule and observing the workers who were hired (and their scores), any worker who was not hired knows that there was no obscure reason why she was not yet hired: it is simply because her score was lower than all those who were in fact hired.

Suppose, however, that the rule that is used is not aggregation independent. Then, a worker who was not hired, by just looking at the set of workers who were hired, may not be able to easily understand why she was not hired, even understanding the rule that was used, because it would also be necessary for her to know the precise sequence of number of workers that were hired in each round.

Non-manipulability

While many times the rules which govern the hiring process are chosen in a way that reduces the ability of managers to make arbitrary choices of whom to hire, they may have freedom in choosing the sequence of hires. For example, instead of hiring four workers in one month, she may choose to hire two workers first and then two additional workers.

If the rule that is used is aggregation independent, different choices of sequences of workers hired will not lead to different sets of workers hired. If the rule is not aggregation independent, however, that may not be the case, and a manager may choose a specific sequence of hiring decisions, which will allow a certain worker to be hired, whereas she would not be, absent the specific sequence chosen. An aggregation independent rule, by definition, is not manipulable by the choice of sequence of hires.

Robustness

The third reason why aggregation independence is a desirable property is that the degree to which the set of workers hired satisfies the objectives represented in the rule is robust to uncertainty or bad planning on the part of the manager in terms of the number of workers that are needed. In other words, assuming that the criterion for choosing workers which is set by the rule represents the desirability of the workers it chooses (for example, it chooses the most qualified set of workers subject to some constraint), an aggregation independent rule will always choose the best set of workers, whether the manager makes hiring decisions all at once or constantly re-evaluates the number of workers to be hired. Aggregation independent rules do not have that problem: managers may hire workers based on demand, and that will not result in a less desirable set of workers hired.

In section 4.4 we show specific examples of how aggregation independence relates to non-manipulability and robustness.

3. Score-based rules

A common way in which workers are selected when hiring by rules is through a scoring of all workers. Using criteria such as written exams, evaluation of diplomas, certificates and experience, workers receive a score (or number of points). These scores become the deciding factor of who to hire: when hiring q workers, hire the q workers with the highest scores from the pool. For a set of workers W, let $\boldsymbol{s_W} = (s_w)_{w \in W}$ be the score profile of workers in W.[3] A natural property for a score-based rule is for it to be **fair**. That is, after any number of rounds, if a worker w was hired and w' was not, then $s_w > s_{w'}$.

Definition 2. A rule φ is **fair** if for any W, A and $\{q_1, \ldots, q_t\}$, $w \in \varphi(W, A, \{q_1, \ldots, q_t\})$ and $w' \notin \varphi(W, A, \{q_1, \ldots, q_t\})$ implies that $s_w > s_{w'}$.

A natural rule for these kinds of problems is what we denote by **sequential priority**. When hiring q workers, it consists of selecting the q workers with the highest score from the pool of workers. If the pool contains less than q workers, then hire all of them. The following remark comes immediately from the definition of the rule.

Remark 1. The sequential priority rule is aggregation independent and fair.

When the selection of workers is based on scores, which is a very common setup, the sequential priority rule gives us all we need: it is fair and aggregation independent.

4. Compositional objectives

It is common for the hiring processes based on rules to combine the use of scores with compositional objectives, such as affirmative action. Typically, the objective is to reserve some of the jobs for workers with a certain characteristic, sometimes those belonging to an ethnic minority or those who possess some type of disability. Denote by M the set of workers who belong to the minority group (that is, $M \subseteq W$) and $m(W')$ be the number of minorities in W'. The affirmative action policy also has a **minority ratio** m, where $0 \leq m \leq 1$, which represents the proportion of hires that should be based on the affirmative action.

[3] Although W is a set, for simplicity of notation we will consider s_W following the order in which the elements of W are written. For example, if we denote $W = \{w_2, w_1, w_3\}$, $s_W = (10, 20, 30)$ implies that worker w_2 has a score of 10.

As argued in section 2, the desirable properties associated with affirmative action should also hold after any number of rounds. Our first requirement is that, when possible, the proportion of selected minorities should be at least m after each round.

Definition 3. A rule φ **respects minority rights** if, for any W and sequence of hires $\{q_1, \ldots, q_t\}$, (i) when $|M| \geq m \times \sum_{i=1}^{t} q_i$ we have $m(\varphi(W, \{q_1, \ldots, q_t\})) / |\varphi(W, \{q_1, \ldots, q_t\})| \geq m$, or (ii) when $|M| < m \times \sum_{i=1}^{t} q_i$ we have $M \subset \varphi(W, \{q_1, \ldots, q_t\})$.

Remark 2. The sequential priority rule does not respect minority rights. In general, fairness is incompatible with respecting minority rights.[4]

Therefore, we define a weaker notion of fairness, which takes into account the minority restriction. A rule is minority fair if, conditional on respecting minority rights, the hiring decision is made based on scores.

Definition 4. A rule φ is **minority fair** if, for any W, $M \subseteq W$ and $\{q_1, \ldots, q_t\}$, where $H = \varphi(W, \{q_1, \ldots, q_t\})$:

(i) for each $w, w' \in W \setminus M$ or $w, w' \in M$, if $w \in H$ and $w' \notin H$, then $s_w > s_{w'}$,

(ii) for each $w \in W \setminus M$ and $w' \in M$, if $s_w < s_{w'}$ and $w \in H$, then $w' \in H$,

(iii) if there is $w \in W \setminus M$ and $w' \in M$ with $s_w > s_{w'}$, $w \notin H$ and $w' \in H$, then $m(H) / |H| \leq m$.

In words, a rule is minority fair if it (i) chooses between workers from the same group (minorities or non-minorities) based on their scores, (ii) does not hire low-scoring non-minorities while higher-scoring minorities are available, and (iii) only hires low-scoring minorities over higher-scoring non-minorities when that is necessary to bring the ratio of minorities closer to m from below.

In the following sections we show that differently from rules that are just based on scores, the design of score-based rules with affirmative action is more challenging. We present rules currently being used in France, Brazil, and Australia, and show that they suffer from different issues. We also show that *minority reserves*, a rule with good properties in a static model, has substantial problems in our setup.

4.1. **Public sector workers in France.** By law, every vacancy in the French public sector must be filled through an *open competition*. When vacancies are announced, a document explaining deadlines, job specifications, and the criteria that will be used to rank the applicants is published. Workers who satisfy some stated requirements then proceed to take writen, oral an/or physical exams. In some cases, diplomas or other certifications can also be used for evaluating the workers. At the end of this process,

[4]Assume that there are three workers: one minority (call him K) and two non-minority (L and V), where the scores are as follows $s_L > s_V > s_K$. If the rule needs to select two workers and m is 0.3, then in order to *respect minority rights*, the rule should select K and L, which is not *fair*, as *fairness* requires L and V to be selected.

the results that all workers had in these tests are combined, in a predetermined way, to produce a ranking over all workers. If the number of vacancies announced was q, then the top q workers are hired. An additional number of workers are put on a "waiting list." These workers on the waiting list may be hired if some of the top q workers reject the job offer or if additional vacancies need to be filled before a new open competition is set.

The French law also establishes that at least 6% of the vacancies should be filled by people with physical or mental disabilities. Instead of incorporating the selection of those workers into the hiring procedure in a unified framework, the institutions instead open, with an unclear regularity, vacancies exclusive for workers who have those disabilities. The hiring of workers over time continues following the same procedure as the open positions described above. However, nothing prevents workers with disabilities from applying for the open positions. In fact, the authorities provide some accommodation for these workers during the selection, such as for example, allowing for extra time to write down the exams. These are meant as an attempt to make up for some disadvantages that those workers have with respect to those without disabilities, and not to give any advantage.

Let $W^* \subseteq W$ and $M^* \subseteq M$ be the set of workers and workers with disabilities who applied and were hired or put onto the waiting list to the open competitions and the competitions reserved for the disabled, respectively. Consider the vacancies announced as $(q_1, q_2, q_3, \ldots, q_r)$. Since workers with disabilities may also apply to the open positions, some of them may apply to both positions. We will therefore allow for the workers with disabilities to make any combination of applications that are possible: only to vacancies for workers with disabilities, only for the open vacancies, or for both. That is, $M^* \subseteq W$ and in general $M^* \cap W^* \neq \emptyset$: all disabled workers are in M and W, and some of them may also be in W^*. Moreover, since these constitute different competitions, the scores among the workers with disabilities and between them and the other workers may be different in both selection processes. Therefore, we denote by s^O_w the score obtained by worker w in the open competition and by s^D_w the score that worker w obtained in the competition for workers with disabilities.[5] The number of vacancies that are open for workers with disabilities, as well as when they are opened, is not determined by any law and is mostly done in an ad-hoc manner. In order to evaluate the consequences of the method used in France in a formal way, however, we will consider two alternative policies, as follows.[6]

Policy 1: This policy consists of first hiring the top q_1 workers from W^* and then adjusting the number of workers in M^* hired in later rounds. For example, say that

[5]The lists s^D_W and s^O_W are also defined accordingly.

[6]We have no evidence that any of these policies constitute actual practice by French institutions, but we believe that they represent the two most natural attempts at satisfying the legal requirements under the current rules.

$q_1 = 100$, but only four workers among the top 100 workers in W^* (with respect to s_W^O) hired have disabilities. Then, considering the objective of hiring at least 6% workers with disabilities, if $q_2 = 50$, then open six vacancies exclusive for workers in M^* (selected with respect to s_W^D) and 44 for those in W^* (selected with respect to s_W^O). As a result, by the end of the second round at least 10 workers with disabilities, or $m \times (q_1 + q_2)$,[7] will be hired.

Policy 2: This policy consists of first hiring $m \times q_1$ from M^* (selected with respect to s_W^D), $(1-m) \times q_1$ workers from W^* (selected with respect to s_W^O) and then adjusting the number of workers in M^* hired in later rounds. For example, say that $q_1 = 100$. Then the policy will result in hiring six workers from M^* and 94 from W^* in the first round. At least 6% of the workers hired would be among those with disability, therefore, but potentially more. Suppose that eight workers with disabilities were hired in the first round and that $q_2 = 50$. Then in the second round, two vacancies exclusive for workers in M^* would be open, and the remaining 48 would be open for all workers in W^*.

Whenever necessary, we will refer to the rules defined by policies 1 and 2 by φ^{F_1} and φ^{F_2}. Since under the French assignment rule each worker may have one or two scores, what constitutes minority fairness is less clear in this context. The example below shows, however, that policy 2 may lead to outcomes that clearly violate the spirit of minority fairness.

Example 1. Let $W^* = \{w_3, w_4, w_5\}$ and $M^* = \{w_1, w_2, w_3\}$, with scores $s_W^O = (50, 40, 30)$ and $s_W^D = (50, 40, 30)$ and $m = 0.5$. If $q = 2$, then $\varphi^{F_2}(\{W^*, M^*\}, q)$ will select $\{w_1, w_3\}$. Worker w_2, however, has a disability and a better score than w_3 in the competition in which both participated.

Notice that if worker w_2 also applied for the open vacancies and in that competition obtained a score that is also better than the one obtained by w_3, she would have been hired instead of w_3. If the relative rankings of the workers in both competitions are different, more intricate violations of the spirit of minority fairness can also take place. If we make the (strong) assumptions that all workers with disabilities apply to both competitions and that the relative ranking between those workers in both competitions are the same, we are able to obtain a clear distinction between both policies, as shown below.

Proposition 1. *Suppose that $M^* \subseteq W^*$ and that for every $w, w' \in M^*$, $s_w^O > s_{w'}^O \iff s_w^D > s_{w'}^D$. Policy 1 of the French assignment rule does not respect minority rights and is not aggregation independent. Policy 2 respects minority rights, is aggregation independent, and minority fair.*

[7]For simplicity, here and in the rest of the main text we will assume that every expression involving numbers of workers or vacancies are integers. In the appendix we relax that restriction, and show that none of the results presented depend on that.

It is important to notice, however, that the result in proposition 1 depends on the **relative rankings of the workers with disabilities being the same in both competitions**, but perhaps most importantly, on **workers with disabilities participating in both competitions**. This is not a minor issue, since these competitions often involve a significant amount of time and effort.

4.2. **Quotas for black public sector job workers in Brazil.** The rules for the hiring of public sector workers in Brazil works essentially in the same way as in France: vacancies are filled with open competitions that result in scores associated with the workers, and workers are hired in each period by following their scores in descending order. Differently from France, however, there is no quota for workers with disabilities, but instead, since 2014, there are quotas for black workers.

The use of racial and income-based quotas has been increasing significantly in many areas of the Brazilian public sector and higher education. At least 50% of the seats in federal universities, for example, are reserved for students who are black, low-income or studied in a public high-school [3]. Many municipalities also employ quotas for black workers in jobs that they offer. One of the most significant recent developments, however, is a law which establishes that 20% of the vacancies offered in each job opening should give priority to black workers.[8]

Differently from France, the quotas for black workers are explicitly incorporated into the hiring process. More specifically, the rule currently used in Brazil (denoted the φ^B rule) works as follows. Let k be a number that is higher than any expected number of hires to be made.

Initial step Workers are partitioned into two groups: (i) Top Minority (TM) and (ii) Others (O). The TM group consists of the highest scoring top $\lceil m \times k \rceil$ workers from M, and O be the top $k - \lceil m \times k \rceil$ workers in $W \backslash TM$.[9] Let $TM^1 = TM$ and $O^1 = O$.

Round $r \geq 1$ The $\lceil m \times q_r \rceil$ top scoring minority workers from TM^r, and the top $q - \lceil m \times q \rceil$ workers from O^r are hired. By removing these workers hired we obtain TM^{r+1} and O^{r+1}.

For the Brazilian law specifically, $m = 0.2$. The example below shows that the Brazilian rule is not minority fair.

Example 2. Let $W = \{w_1, w_2, w_3, w_4\}$, $M = \{w_1, w_2\}$, and $s_W = (100, 90, 80, 50)$. Let $q = 2$, $m = 0.5$ and $k = 4$. Then $TM^1 = \{w_1, w_2\}$ and $O^1 = \{w_3, w_4\}$. The Brazilian rule states that, when hiring two workers, the top worker from TM^1 and the

[8] Lei N. 12.990, de 9 de junho de 2014.

[9] If there are not enough minority workers, the remaining positions are filled with the top non-minority workers.

top from O^1 should be hired. Therefore, $\varphi^B(W,q) = \{w_1, w_3\}$. Since $w_2 \notin \varphi^B(W,q)$ and $s_{w_2} > s_{w_3}$, the Brazilian rule is not *minority fair*.[10]

Notice that in this example, worker w_2, who is part of the minority, has a higher score than w_3, who is not a minority. Worker w_3 is hired, while w_2 is not. Given that the affirmative action rules were designed with the intent of increasing the access that minorities have to these jobs, this type of lack of fairness is especially undesirable, since if the hiring process was purely merit-based, worker w_2 would have been hired.

[3] describe the implementation of affirmative action in the admission to Brazilian public universities. There, as here, the problems arise from the fact that positions (in that case seats) and workers are partitioned between those reserved for minorities and the open positions. Differently from there, however, unfair outcomes may not be prevented by students even if they strategically manipulate their minority status. In the example above, even if w_2 applied as a non-minority he would not be hired.

Proposition 2. *The Brazilian rule is aggregation independent and respects minority rights. However, it is not minority fair.*

4.3. **Gender balance in the hiring of firefighters in New South Wales.** The hiring of firefighters in the Australian province of New South Wales attempts to achieve a gender-balanced workforce by following a simple rule:

> "Candidates who have successfully progressed through the recruitment stages may then be offered a place in the Firefighter Recruitment training program. Written offers of employment will be made to an equal number of the most meritorious male and female candidates based on performance at interview and the other components of the recruitment process combines."[11]

We denote this rule by *NSW rule*, or φ^{NSW}. Although not stated explicitly in the institution's website, we will assume that if there are not enough individuals from some gender, the remaining hirings will be made among those candidates available, based on their scores. Also, to avoid results that rely simply on whether q is odd or even, we assume that it is always even. The example below shows the problems involved in that rule:

Example 3. Let $W = \{w_1, w_2, w_3, w_4\}$, and $W^F = \{w_1, w_2\}$ and $W^M = \{w_3, w_4\}$ be the set of female and male workers, respectively. Suppose that the scores are $s_W =$

[10]One may conjecture that the scenario above is very unexpected, since the affirmative action law must have been enacted in response to minority workers not being hired based solely on scores. As shown in [3], however, this conjecture may be misleading. For example, even if the average score obtained by minority workers is lower, one can have situations in which the preferences of the higher achieving minority workers are correlated, leading to the top minority workers in the entire population applying to a specific job.

[11]Source: Fire & Rescue NSW (https://www.fire.nsw.gov.au/page.php?id=9126)

$(100, 90, 80, 50)$. Let $q = 2$. Then $\varphi^{NSW}(W, q) = \{w_1, w_3\}$. Since w_2 is not hired but $s_{w_2} > s_{w_3}$, the NSW rule is not fair. Moreover, it is easy to see that if either gender is considered a minority, the rule is also not minority fair.

The result below summarizes the properties of the NSW rule.

Proposition 3. *The NSW rule is aggregation independent but not fair. If one of the genders is deemed as a minority, then it respects minority rights but is not minority fair.*

4.4. **Sequential use of minority reserves.** Next, we consider what may be a natural rule to be used. In each period in which there are vacancies to be filled, the institution uses a choice procedure generated by reserves [10, 7], with a proportion m of the vacancies reserved for minority workers.

Given a set of workers W, of minorities $M \subseteq W$, a number of reserved positions q^m and of hires q, a choice generated by reserves consists of hiring the top $\min\{q^m, |M|\}$ workers from M and then filling the remaining $q - \min\{q^m, |M|\}$ positions with the top workers in W still available. In a static setting, this procedure is shown to have desirable fairness and efficiency properties, while satisfying the distributional objectives [10]. We denote the sequential use of minority reserves rule by φ^{SM}.

In our setting, therefore, the sequential use of minority reserves rule consists of, in round r, hiring q_r workers, reserving $m \times q_r$ of them for minorities.

Proposition 4. *The sequential use of minority reserves respects minority rights. However it is neither minority fair nor aggregation independent.*

The next example shows the problems associated with this rule.

Example 4. Let $W = \{w_1, w_2, w_3, w_4, w_5\}$, $M = \{w_1, w_2, w_5\}$, and $s_W = (100, 90, 80, 50, 20)$. Let $r = 2$, $q_1 = q_2 = 2$ and $m = 0.5$. In the first round, the top worker from M and the top from $W \setminus \{w_1\}$ are hired, that is, $\{w_1, w_2\}$. In the second round, the pools of remaining workers are $W^2 = \{w_3, w_4, w_5\}$ and $M^2 = \{w_5\}$. The top worker from M^2, that is, $\{w_5\}$, and the top from $W^2 \setminus \{w_5\}$ are hired. Therefore, $\{w_3, w_5\}$ are hired in the second round and $\varphi^{SM}(W, \{q_1, q_2\}) = \{w_1, w_2, w_3, w_5\}$.

Now consider the case where $q = q_1 + q_2 = 4$. Then in the first and unique round, the two top workers from M, $\{w_1, w_2\}$, and the top two workers from $W \setminus \{w_1, w_2\}$ are hired, that is, $\{w_3, w_4\}$. Therefore, $\varphi^{SM}(W, q_1 + q_2) = \{w_1, w_2, w_3, w_4\}$. Therefore, the φ^{SM} rule is *not aggregation independent*. Also, note that $w_4 \in \varphi^{SM}(W, \{q_1, q_2\})$, $w_5 \in \varphi^{SM}(W, \{q_1, q_2\})$ and $s_{w_4} = 50 > 20 = s_{w_5}$ while $m\left(\varphi^{SM}(W, \{q_1, q_2\})\right) / \left|\varphi^{SM}(W, \{q_1, q_2\})\right| = 0.75 > 0.5 = m$, implying that the φ^{SM} rule is *not minority fair.*

The sequential use of minority reserves rule is a good rule for providing examples of the problems associated with rules that are not aggregation independent. First, consider the issue of **manipulability**. Take example 4 above and suppose that the manager needs to hire four workers and have a preference for hiring worker w_5. If she hires the four workers that she needs, all at once, w_5 would not be hired. If, instead, she chooses to hire first two workers, and then later two more workers, w_5 will be hired. That is, by choosing a sequence of hires strategically, the manager is able to hire the person she wanted.

Next, we show that the lack of aggregation independence may lead to the hiring of a group of workers who are not in line with some common objectives of desirability, (the issue of **robustness**, as described in section 2). To see how this can be a problem, consider the example below:

Example 5. Let $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$, $M = \{w_5, w_6, w_7, w_8\}$, $s_W = (100, 90, 80, 70, 60, 50, 40, 30)$ and $m = 0.5$.

If workers are hired in four rounds, where $q_1 = q_2 = q_3 = q_4 = 1$, the set of workers hired will be $\{w_5, w_6, w_7, w_8\}$. If, on the other hand, workers are hired all at once, with $q_1 = 4$, the set of workers hired will be $\{w_1, w_2, w_5, w_6\}$

Assuming that the scores are a good representation of the degree of desirability of a worker for a task, the example above shows that a lack of planning could lead to hiring a set of workers that are substantially less qualified.

5. Sequential adjusted minority reserves

Now we present a new rule, sequential adjusted minority reserves, denoted by φ^{SA}. It consists of the sequential minority reserves rule in which the number of vacancies reserved for minorities is adjusted in response to hires made in previous rounds. More specifically, the rule works as follows:[12]

Round 1 Let $m^1 = m$, $M^1 = M$ and $W^1 = W$. The top $m^1 \times q_1$ workers from M^1 are hired. Denoted those workers by A^*. Additionally, the top $(1 - m^1) \times q_1$ workers from $W^1 \backslash A^*$ are hired. Let M^2 be the workers in M^1 who were not yet hired, and W^2 be the workers in W^1 who were not yet hired.

Round $r \geq 1$ Let $A^r = \varphi^{SA}(W, \{q_1, \ldots, q_{r-1}\})$ and $m^r = \max\left\{m - \frac{m(A^r)}{\sum_{i=1}^{r} q^i}, 0\right\}$. The $m^r \times q_r$ top scoring minority workers in M^{r-1} are hired. Denote those workers by A^*. Additionally, the top $(1 - m^r) \times q_r$ workers from $W^r \backslash A^*$ are hired. Let M^{r+1} be the workers in M^r who were not yet hired, and W^{r+1} be the workers in W^r who were not yet hired.

Differently from most previous rules, the sequential adjusted minority reserves adapts the set of workers hired according to those who were hired in previous rounds. This

[12]For simplicity, the description below assumes that the number of workers in M and W is large enough so that in every round there is a sufficient number of them to be hired. A more general description can be found in the appendix.

makes sense: if we do not take into account, for example, that after the last round the number of minority workers greatly exceeded the minimum required, some high-scoring non-minority workers may not be hired, leading to a violation of minority fairness. In fact, the theorem below shows that this is the only way of achieving these objectives.

Theorem 1. *The sequential adjusted minority reserves is the unique rule that is minority fair and respects minority rights.*

Moreover, the sequential adjusted minority reserves is aggregation independent, making it ideal for a sequential hiring from a pool of workers.

Proposition 5. *The sequential adjusted minority reserves rule is aggregation independent.*

6. Multiple institutions

In many cases, a pool of workers is shared between multiple institutions or locations. In the hiring process for the Brazilian federal police, for example, workers may be allocated to different locations.[13] In the selection process for the New Zealand police, the candidate's preference is also taken into account when deciding which district a worker who will be hired from the pool will go to:

> "The candidate pool is not a waiting list. The strongest candidates are always chosen according to the needs and priorities of the districts. The time it takes to get called up to college depends on your individual strengths and the constabulary recruitment requirements in your preferred districts. (...) We will look to place you into your preferred district but you may also be given the option to work in another district where recruits are needed most."[14]

In this section we evaluate how the fact that workers may be hired by more than one institution affects the attainability of basic desirable properties. Now, in addition to the set of workers W, there is a set of institutions $I = \{i_1, \ldots, i_\ell\}$. Institutions make sequences of hires, and there is no simultaneity in their hires: in each round only one institution may hire workers. Therefore, when we describe a round we now must determine not only how many workers are hired, but also which institution those workers will be assigned to. Some additional notation will be necessary. A **matching** μ is a function from $I \cup W$ to subsets of $I \cup W$ such that:

- $\mu(w) \in I \cup \{\emptyset\}$ and $|\mu(w)| = 1$ for every worker i,[15]
- $\mu(i) \subseteq W$ for every institution i,

[13]Source: Brazilian Department of Federal Police.
[14]Source: New Zealand Police (https://www.newcops.co.nz/recruitment-process/candidate-pool), accessed in March 8th 2018.
[15]We abuse notation and consider $\mu(w)$ as an element of I, instead of a set with an element of I.

- $\mu(w) = i$ if and only if $w \in \mu(i)$.

At the end of each round $r \geq 1$, we define the **matching** of workers to institutions as a function μ^r.

A sequence of hires is a list of pairs (i, q), where i is an institution and q is the number of workers hired. A sequence of hires $q_I^r = ((i_1, 3), (i_3, 2), (i_1, 2))$, for example, represents the case in which in the first round institution i_1 hires three workers, in the second round institution i_3 hires two workers, and then in the third round institution i_1 hires two workers. A rule φ therefore, can be generalized to produce matchings instead of allocations. We will use the notation φ_i for the value of $\varphi(\cdot)(i)$. The example below clarifies these points.

Example 6. Consider a set of workers $W = \{w_1, w_2, w_3, w_4, w_5\}$ with scores $s_W = (100, 90, 80, 50, 20)$, a set of institutions $I = \{i_1, i_2, i_3\}$ and let φ be a rule that, in any round, matches the highest scoring workers to the institution in that round. Then if $q_I^r = ((i_1, 1), (i_3, 2), (i_1, 1))$, the matchings μ^1, μ^2 and μ^3 produced at the end of each round are:

$$\mu^1 = \begin{pmatrix} i_1 & i_2 & i_3 \\ w_1 & \emptyset & \emptyset \end{pmatrix} \quad \mu^2 = \begin{pmatrix} i_1 & i_2 & i_3 \\ w_1 & \emptyset & \{w_2, w_3\} \end{pmatrix} \quad \mu^3 = \begin{pmatrix} i_1 & i_2 & i_3 \\ \{w_1, w_4\} & \emptyset & \{w_2, w_3\} \end{pmatrix}$$

We will consider two properties for rules when there are multiple institutions. The first is related to the desirability of workers.

Definition 5. A rule φ satisfies **common top** if there exists a worker $w^* \in W$ such that, for every institution $i \in I$ and $q > 0$, $w^* \in \varphi_i(W, (i, q))$.

In words, common top requires that there is at least one worker that, whenever available, all institutions would hire.

Next, we consider a weak notion of consistency across the hirings made by the institutions.

Definition 6. A rule φ satisfies **permutation independence** if for any sequence of hires q_I^r and any permutation of its elements $P(q_I^r)$, $\bigcup_{i \in I} \varphi_i(W, q_I^r) = \bigcup_{i \in I} \varphi_i(W, P(q_I^r))$.∎

Permutation independence, therefore, simply requires that the set of workers hired, regardless of where, should not change if we adjust the order of hirings.

The family of rules that we will use in our next result is very simple but also very restrictive. Let q_I^r be any sequence of hires. A rule φ is **single priority** if there exists a strict ranking $\succ$ of the workers in W such that when q_I^r is any sequence of hires,

$$\varphi_i(W, (q_I^r, (i, q))) = \varphi_i(W, q_I^r) \cup \max_{\succ}^{q} W \backslash \varphi_i(W, q_I^r)$$

Where $\max_{\succ}^{q} X$ is the set with the top q elements of X with respect to the ordering $\succ$. In words, a rule is single priority if all hirings from all institutions consist of hiring the top workers, among the remaining ones, when all of these institutions share a common ranking.

The result below shows that, for a wide range of applications, having multiple institutions is incompatible with most objectives a policymaker may have.

Theorem 2. *A rule satisfies common top, aggregation independence and permutation independence if and only if it is a single priority rule.*

Theorem 2 is fundamentally a negative result. It implies that, for the most part, distributional objectives, such as affirmative action, though very common in practice, are unattainable unless we give up on this reasonably weak concept of independence.

7. Conclusion

In this paper, we evaluate a hiring method that is widely used around the world, especially for public sector jobs, where institutions select their workers over time from a pool of eligible workers. While the simple and natural rule of sequential priority satisfies all desirable characteristics, the addition of compositional objectives such as affirmative action policies increases the complexity of the procedures. In fact, we show that the rules being used in practical hiring processes, as well as the direct application of minority reserves, fail fairness or aggregation independence. When the compositional objectives can be modeled as affirmative action for minorities, the sequential adjusted minority reserves, which we introduced, is therefore the unique solution that satisfies those desirable properties.

If multiple institutions hire from the same pool of applicants, however, we show that the space for compositional objectives, or even distinct hiring criteria between institutions, is highly restricted when a minimal requirement of independence is imposed.

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Appendix

Formal descriptions of the rules.

Sequential Priority (SP rule).

For each $(W, A, \{q_1, \dots q_r\})$, each round $a \leq r$, the highest scoring workers are selected.

Round 1: Let $W_1 = W$ and.$M_1 = M \cap W_1$. The highest scoring q_1 workers in W_1 are selected. Let A_1, be the set of selected workers, where for each $w \in A_1$ and each $w' \in W_1 \setminus A_1$ we have $s_w > s_{w'}$, and $|A_1| = q_1$.

Round $k = 2, 3, \dots, r$: Let $W_k = W_{k-1} \setminus A_{k-1}$ and $M_k = M \cap W_k$. The highest scoring q_k workers in W_k are selected. Let A_k be the set of selected workers, where for each $w \in A_k$ and each $w' \in W_k \setminus A_k$ we have $s_w > s_{w'}$, and $|A_k| = q_k$.

The assignment selected by SP rule is
$$\varphi^{SP}(W, \{q_1, \dots, q_r\}) = \bigcup_{a \leq r} A_a.$$

Sequential Adjusted Minority Reserves (SAM rule).

For each $(W, A, \{q_1, \dots q_r\})$, we have the following two steps

Round 1:

Step 1.1: Let $W_{1,1} = W$, $M_{1,1} = M \cap W_{1,1}$ and $q_{1,1} = \lceil m \times q_1 \rceil$. The highest scoring $\min\{q_{1,1}, |M_{1,1}|\}$ minority workers in $W_{1,1}$ are selected. Let $A_{1,1}$ be the set of selected workers, where $A_{1,1} \subseteq M_{1,1}$. For each $w \in A_{1,1}$ and each $w' \in M_{1,1} \setminus A_{1,1}$, we have $s_w > s_{w'}$, and $|A_{1,1}| = \min\{q_{1,1}, |M_{1,1}|\}$.

Step 1.2: Let $W_{1,2} = W_{1,1} \setminus A_{1,1}$, $M_{1,2} = M \cap W_{1,2}$ and $q_{1,2} = q_1 - |A_{1,1}|$. The highest scoring $q_{1,2}$ workers in $W_{1,2}$ are selected. Let $A_{1,2}$ be the set of selected workers, where for each $w \in A_{1,2}$ and each $w' \in W_{1,2} \setminus A_{1,2}$ we have $s_w > s_{w'}$, and $|A_{1,2}| = q_{1,2}$.

Round $k = 2, 3, \dots, r$:

Step k.1: Let $W_{k,1} = W_{k-1,2} \setminus A_{k-1,2}$, $M_{k,1} = M \cap (W_{k-1,2} \setminus A_{k-1,2})$ and $q_{k,1} = \lceil \min\{\max\{m - \frac{m(A_{1,2})+\ldots+m(A_{k-1,2})}{q_k}, 0\} \times q_k, |M_{k,1}|\} \rceil$. The highest scoring $q_{k,1}$ minority workers in $W_{k,1}$ are selected. Let $A_{k,1}$ be the set of selected workers, where $A_{k,1} \subseteq M_{k,1}$, for each $w \in A_{k,1}$ and each $w' \in M_{k,1} \setminus A_{k,1}$ we have $s_w > s_{w'}$, and $|A_{k,1}| = q_{k,1}$.

Step k.2: Let $W_{k,2} = W_{k,1} \setminus A_{k,1}$, $M_{k,2} = M \cap W_{k,2}$ and $q_{k,2} = q_k - |A_{k,1}|$. The highest scoring $q_{k,2}$ workers are selected from $W_{k,2}$. Let $A_{k,2}$ be the set of selected workers, where for each $w \in A_{k,2}$ and each $w' \in W_{k,2} \setminus A_{k,2}$ we have $s_w > s_{w'}$, and $|A_{k,2}| = q_{k,2}$.

The assignment selected by the SAM rule is

$$\varphi^{SAM}(W, A, \{q_1, \dots, q_r\}) = \bigcup_{\substack{a \le r \\ i \in \{1,2\}}} A_a^i.$$

Sequential use of minority reserves (SM rule).

Within each round $a \le r$ we have two steps.

Round 1:

Step 1.1: Let $W_{1,1} = W$, $M_{1,1} = M \cap W_{1,1}$ and $q_{1,1} = \lceil m \times q_1 \rceil$. The highest scoring $\min\{q_{1,1}, |M_{1,1}|\}$ minority workers are selected. Let $A_{1,1}$ be the set of selected workers, where $A_{1,1} \subseteq M_{1,1}$. For each $w \in A_{1,1}$ and each $w' \in W_{1,1} \setminus A_{1,1}$ we have $s_w > s_{w'}$, and $|A_{1,1}| = \min\{q_{1,1}, |M_{1,1}|\}$.

Step 1.2: Let $W_{1,2} = W \setminus A_{1,1}$, $M_{1,2} = M \cap W_{1,2}$ and $q_{1,2} = q_1 - |A_{1,1}|$. The highest scoring $q_{1,2}$ workers are selected. Let $A_{1,2}$ be the set of selected workers, where for each $w \in A_{1,2}$ and each $w' \in W_{1,2} \setminus A_{1,2}$ we have $s_w > s_{w'}$, and $|A_{1,2}| = q_{1,2}$.

Round $k = 2, 3, \dots, r$:

Step k.1: Let $W_{k,1} = W_{k-1,2} \setminus A_{k-1,2}$, $M_{k,1} = M \cap (W_{k-1,2} \setminus A_{k-1,2})$ and $q_{k,1} = \lceil m \times q_k \rceil$. The highest scoring $\min\{q_{k,1}, |M_{k,1}|\}$ minority workers are selected. Let $A_{k,1}$ be the set of selected workers, where $A_{k,1} \subseteq M_{k,1}$. For each $w \in A_{k,1}$ and each $w' \in W_{k,1} \setminus A_{k,1}$ we have $s_w > s_{w'}$, and $|A_{k,1}| = \min\{q_{k,1}, |M_{k,1}|\}$.

Step k.2: Let $W_{k,2} = W \setminus A_{k,1}$, $M_{k,2} = M \cap W_{k,2}$ and $q_{k,2} = q_k - |A_{k,1}|$. The highest scoring $q_{k,2}$ workers are selected. Let $A_{k,2}$ be the set of selected workers, where for each $w \in A_{k,2}$ and each $w' \in W_{k,2} \setminus A_{k,2}$ we have $s_w > s_{w'}$, and $|A_{k,2}| = q_{k,2}$.

The assignment produced by the SM rule is $\varphi^{SM}(W, q^r, r) = \bigcup_{\substack{a \le r \\ i \in \{1,2\}}} A_{a,i}$.

Brazilian assignment rule (B rule).

Let $\pi = (W, M, s_W, q^r, r)$ be a problem. The rule identifies a large number k (which is larger than the total number of vacancies to be filled). Then two groups are identified: (i) the top $k \times m$ minority workers: $TM \subseteq M$ with $|TM| = \lceil k \times m \rceil$ such that for each $w \in TM$ and each $w' \in M \setminus TM$, we have $s_w > s_{w'}$ and (ii) the top $k(1-m)$ workers among those who were not chosen in (i), that is: $O \subseteq W \setminus TM$ such that $|O| = \lfloor k(1-m) \rfloor$ and for each $w \in O$ and$w' \in W \setminus (O \cup TM)$, we have $s_w > s_{w'}$. Within each round $a \le r$, we have two steps.

Round 1:

Step 1.1: Let $O_{1,1} = O$, $TM_{1,1} = TM$ and $q_{1,1} = \lceil m \times q_1 \rceil$. The highest scoring $\min\{q_{1,1}, |TM_{1,1}|\}$ minority workers are selected from $TM_{1,1}$. Let$A_{1,1}$ be the set of selected workers, where $A_{1,1} \subseteq TM_{1,1}$. For each $w \in A_{1,1}$ and each $w' \in TM_{1,1} \setminus A_{1,1}$ we have $s_w > s_{w'}$, and $|A_{1,1}| = \min\{q_{1,1}, |TM_{1,1}|\}$.

Step 1.2: Let $O_{1,2} = O_{1,1}$, $TM_{1,2} = TM \setminus A_{1,1}$ and $q_{1,2} = q_1 - |A_{1,1}|$. The highest scoring $q_{1,2}$ workers are selected from $O_{1,2}$. Let $A_{1,2}$ be the set of selected workers, where for each $w \in A_{1,2}$ and each $w' \in O_{1,2} \setminus A_{1,2}$ we have $s_w > s_{w'}$, and $|A_{1,2}| = q_{1,2}$.

Round*$\boldsymbol{k = 2, 3, \ldots, r}$*:

Step k.1: Let $O_{k,1} = O_{k-1,2} \setminus A_{k-1,2}$, $TM_{k,1} = TM_{k-1,2}$ and $q_{k,1} = \lceil m \times q_k \rceil$. The highest scoring $\min\{q_{k,1}, |TM_{k,1}|\}$ minority workers are selected from $TM_{k,1}$. Let $A_{k,1}$ be the set of selected workers, where $A_{k,1} \subseteq TM_{k,1}$. For each $w \in A_{k,1}$ and each $w' \in TM_{k,1} \setminus A_{k,1}$ we have $s_w > s_{w'}$, and $|A_{k,1}| = \min\{q_{k,1}, |TM_{k,1}|\}$.

Step k.2: Let $O_{k,2} = O_{k,1}$, $TM_{k,2} = TM \setminus A_{k,1}$ and $q_{k,2} = q_k - |A_{k,1}|$. The highest scoring $q_{k,2}$ workers are selected from $O_{k,2}$. Let$A_{k,2}$ be the set of selected workers, where for each $w \in A_{k,2}$ and each $w' \in O_{k,2} \setminus A_{k,2}$ we have $s_w > s_{w'}$, and $|A_{k,2}| = q_{k,2}$.

The assignment produced by the B rule is $\varphi^{B}(W, q^r) = \bigcup_{\substack{a \le r \\ i \in \{1,2\}}} A_{a,i}$.

French assignment rule (F rule).

Let m be the target ratio of people with disabilities.

Round 1:

Policy 1: Let $W_{1,1} = W$, $M_{1,1} = M \cap W_{1,1}$. The highest scoring $\min\{q_{1,1}, |W_{1,1}|\}$ workers, with respect to s_W^O, are selected from $W_{1,1}$. Let $A_{1,1}$ be the set of selected workers, where $A_{1,1} \subseteq W_{1,1}$.

Policy 2: Let $W_{1,1} = W$, $M_{1,1} = M$. The highest scoring $\min\{\lfloor(1-m) \times q_{1,1}\rfloor, |M_{1,1}|\}$ workers, with respect to s_W^D, are selected from $M_{1,1}$. Let $A_{1,1}$ be the set of workers selected in this step. Then, the highest scoring $\min\{\lceil m \times q_{1,1}\rceil, |W_{1,1} \setminus A_{1,1}|\}$ workers, with respect to s_W^O, are selected from $W_{1,1} \setminus A_{1,1}$. Let $A_{1,2}$ be the set of selected workers in this step, and let $A_1 = A_{1,1} \cup A_{1,2}$.

Round $k = 2, 3, \ldots, r$:

Step k.1: Let $W_{k,1} = W_{k-1,2} \setminus A_{k-1,2}$, $TA_{k,1} = \bigcup_{i=1}^{k-1} A_i$. Let $q_{k,1} = min\left\{max\left\{m \times \left(\sum_{i=1}^{k} q_i\right) - m\left(TA_{k,1}\right), 0\right\}, |M_{k,1}|\right\}$. The highest scoring $q_{k,1}$ workers, with respect to s_W^D, are selected from $M_{k,1}$. Let $A_{k,1}$ be the set of workers selected in this step.

Step k.2: Let $W_{k,2} = W_{k,1} \setminus A_{k,1}$, and $q_{k,2} = q_k - |A_{k,1}|$. The highest scoring $q_{k,2}$ workers, with respect to s_W^O, are selected from $W_{k,2}$. Let $A_{k,2}$ be the set of selected workers, $A_k = A_{k,1} \cup A_{k,2}$ and $TA_{k,2} = TA_{k,1} \cup A_{k,2}$.

The assignment produced by the F rule is $\varphi^F(W, q^r, r) = \bigcup_{\substack{a \leq r \\ i \in \{1,2\}}} A_{a,i}$.

Proofs.

Proposition 1. Let $W^* = \{w_1, w_2, w_3, w_4, w_5\}$ with scores $s_W = (50, 40, 30, 20, 10)$. For simplicity we will use $m = 0.5$.

Consider first the case $M^* = \{w_3, w_4\}$. If $q = 2$, $\varphi^{F_1}(\{W^*, M^*\}, q) = \{w_1, w_2\}$, which fails to satisfy minority rights.

Consider now the case $M^* = \{w_4, w_5\}$. Consider two possibilities: $q_1 = q_2 = 2$ and $q = 4$. Then $\varphi^{F_1}(\{W^*, M^*\}, \{q_1, q_2\}) = \{w_1, w_2, w_4, w_5\}$ but $\varphi^{F_1}(\{W^*, M^*\}, q) = \{w_1, w_2, w_3, w_4\}$, a violation of *aggregation independence.*

It is easy to see that the rule that results from policy 2, under the given assumptions, is equivalent to the sequential adjusted minority reserves rule. Therefore, Policy 2 of the French assignment rule respects minority rights, is aggregation independent, and is minority fair.

Proposition 2. We will show that the Brazilian rule respects minority rights and is aggregation independent, but fails to be minority fair.

By assumption, no more than k workers may be hired in total. Therefore, for any q workers to be hired in any given round there should be at least $\lceil q \times m\rceil$ minority workers in TM and $q - \lceil q \times m\rceil$ workers in O. As a result, the Brazilian rule acts as two parallel sequential priority rules: one in TM and one in O. Therefore, the combination of both is evidently aggregation independent. Next, notice that again because of the assumption on the value of k, $|M| \geq m \times \sum_{i=1}^{t} q_i$. Moreover, since for any $q \in q_1, \ldots, q_t$ there are at least $\lceil q \times m \rceil$ minority workers in TM, $m(\varphi(W, \{q_1, \ldots, q_t\})) \geq m \times \sum q_i$ and by assumption on k, $|\varphi(W, \{q_1, \ldots, q_t\})| = \sum q_i$ therefore $m(\varphi(W, \{q_1, \ldots, q_t\})) / |\varphi(W, \{q_1, \ldots, q_t\})| \geq m$, implying that the Brazilian rule respects minority rights. Finally, example 2 shows that the rule is not minority fair.

Proposition 3. Example 3 shows that the NSW rule is neither fair nor minority fair. Moreover, since for our results we assume that the number of men and women are always large enough, the NSW consists of two parallel sequential priority hirings (one for male, the other for female workers), and therefore satisfies aggregation independence. Finally, it respects minority rights, since the number of male and female workers hired is always the same.

Proposition 4. Example 4 shows that the sequential use of minority reserves is neither aggregation independent nor fair. To see that it respects minority rights, notice that every time q workers are hired, **at least** $m \times q$ minority workers are among them. As a result, a proportion of at least m of the workers hired, at any point, is in M and therefore the rule respects minority rights.

Theorem 1. By definition, the SAM rule *respects minority rights*, as at step $k.1$ of each round k, rule selects minority workers to satisfy the minimum requirement up to that round. Note that when there are not enough minority workers, SAM selects all the available minority workers.

Now, we show that the rule is minority fair.

Let $A \equiv \varphi^{SAM}(W, \{q_1, \ldots, q_r\})$ be the selection made for the problem. We want to show that (i) for each $w, w' \in W \setminus M$, if $w \in A$ and $w' \notin A$, then $s_w > s_{w'}$, (ii) for each $w, w' \in M$, if $w \in A$ and $w' \notin A$, then $s_w > s_{w'}$. (iii) for each $w \in W \setminus M$ and $w' \in M$, if $s_w < s_{w'}$ and $w \in A$, then $w' \in A$, (iv) if there is $w \in W \setminus M$ and $w' \in M$ with $s_w > s_{w'}$, $w \notin A$ and $w' \in A$, then $m(A)/n(A) \leq m$.

First note that cases (i), (ii) and (iii) hold trivially as at step $k.1$ of each round k, the rule selects the highest scoring workers in M, and in step $k.2$ it selects the highest scoring workers.

Suppose, for contradiction, that there is $w \in W \setminus M$ and $w' \in M$ with $s_w > s_{w'}$, $w \notin A$ and $w' \in A$, but $m(A)/n(A) > m$. Note that w' cannot be selected at step $k.2$ of any round k, as w would have been selected as well. The only case in which candidate w' is selected is during step $\ell.1$ of some round ℓ. Since $s_w > s_{w'}$, $w \notin A$ and $w' \in A$, then $|top_q(W) \cap M| < m \times q$, where $q = \sum_{a \leq r} q_a$.[16] Thus, at step $r.1$ of the last round r,

[16] That is, the only way to hire a minority worker with a lower score and not the non-minority with a higher score, is to satisfy the minority requirements. As we mentioned earlier, worker w' is hired during step $\ell.1$ of some round ℓ, where selection occurs among minorities only.

a selection is made so that$|(\bigcup_{\substack{a<r\\ i\in\{1,2\}}} A_a^i) \cup A_k^1| = m \times q$. Since $|top_q(W) \cap M| < m \times q$, we have $A_r^2 \cap M = \emptyset$. Thus, we obtain $A \cap M = m \times q$ which contradicts our assumption.

Now, we prove equivalence.

Let $A \equiv \varphi^{SAM}(W, \{q_1, \ldots, q_r\})$ and $A' = \varphi(W, q^r, r)$. Suppose, for contradiction, there is $w \in A$ such that $w \notin A'$. Since $|A| = |A'|$, there is $w' \in A'$ such that $w' \notin A$. We have several cases.

Case 1: $\boldsymbol{w, w' \in W \setminus M}$. If $s_w > s_{w'}$, then since φ is *minority fair*, $w' \in A'$ implies that $w \in A'$ as well, which contradicts our assumption. If reversely, $s_w < s_{w'}$, then since SAM is *minority fair*, $w \in A$ implies that $w' \in A$ as well, which contradicts our assumption.

Case 2: $\boldsymbol{w, w' \in M}$. If $s_w > s_{w'}$, then since φ is *minority fair*, $w' \in A'$ implies that $w \in A'$ as well, which contradicts our assumption. If reversely, $s_w < s_{w'}$, then since SAM is *minority fair*, $w \in A$ implies that $w' \in A$ as well, which contradicts our assumption.

Case 3: $\boldsymbol{w' \in W \setminus M}$ and $\boldsymbol{w \in M}$. If $s_w > s_{w'}$, then since φ is *minority fair*, $w' \in A'$ implies that $w \in A'$ as well, which contradicts our assumption. Suppose instead that $s_w < s_{w'}$. Since $w \in A$ and $w' \notin A$ we have either (i) $|M| < \lceil m \times q \rceil$ and thus $|M \cap A| < \lceil m \times q \rceil$, where $q = \sum_{a \leq r} q_a$, or (ii) $|M| \geq \lceil m \times q \rceil$ and thus $|M \cap A| = \lceil m \times q \rceil$. If we have (i), then since SAM *respects minority rights*, we have $M \subseteq A$, and since φ also *respects minority rights* we have $M \subseteq A'$ which implies $w \in A'$, which contradicts our assumption. Reversely, let (ii) be the case. Then since $|M| \geq \lceil m \times q \rceil$ and since φ *respects minority rights*, the fact that $w \notin A'$ implies that there is $w'' \in A' \cap M$ such that $w'' \notin A$. Since $w'' \notin A$, the *minority fairness* of SAM implies that $s_{w''} < s_w$. Then, by *minority fairness* of φ, the fact that $w'' \in A'$ implies that $w \in A'$, which contradicts our assumption.

Case 4: $\boldsymbol{w \in W \setminus M}$ and $\boldsymbol{w' \in M}$. If $s_{w'} > s_w$, then since SAM is *minority fair*, $w \in A$ implies that $w' \in A$ as well, which contradicts our assumption. Suppose $s_w > s_{w'}$. Since $w' \in A'$ and $w \notin A'$ we have either (i) $|M| < \lceil m \times q \rceil$ and thus $|M \cap A'| < \lceil m \times q \rceil$, where $q = \sum_{a \leq r} q_a$, or (ii) $|M| \geq \lceil m \times q \rceil$ and thus $|M \cap A'| = \lceil m \times q \rceil$. If we have (i), then since φ *respects minority rights* we have $M \subseteq A'$, and since SAM also *respects minority rights* we have $M \subseteq A$ which implies $w' \in A$, which contradicts our assumption. Reversely, let (ii) be the case. Then, since $|M| \geq \lceil m \times q \rceil$ and since SAM*respects minority rights*, *minority fairness* implies that $|A \cap M| = |A' \cap M|$. Then, the fact that $w' \notin A$ implies that there is $w'' \in A \cap M$ such that $w'' \notin A'$. Since $w'' \notin A'$, *minority fairness* of φ implies that $s_{w''} < s_{w'}$. Then, by *minority fairness* of SAM, the fact that $w'' \in A$ implies that $w' \in A$, which contradicts our assumption.

Proposition 5. Let W be the set of workers. First, note that when $|M| < \lceil m \times q \rceil$, $M \subset \varphi^{SAM}(W, \{q_1, \ldots, q_r\})$ and $M \subset \varphi^{SAM}(W, \sum_{i=1}^{r} q_i)$. Since $|\varphi^{SAM}(W, \{q_1, \ldots, q_r\})| =$

$|\varphi^{SAM}(W, \sum_{i=1}^{r} q_i)|$ and SAM is *minority fair*, we have $M \subset \varphi^{SAM}(W, \{q_1, \ldots, q_r\})$ and thus $\varphi^{SAM}(W, M, s_W, q^r, r) = \varphi^{SAM}(W, \sum_{i=1}^{r} q_i)$.

Now let $|M| \geq \lceil m \times q \rceil$. Suppose, for contradiction, $\varphi^{SAM}(W, M, s_W, q^r, r) \neq \varphi^{SAM}(W, \sum_{i=1}^{r} q_i)$.

Let $A' = \varphi^{SAM}(W, M, s_W, q^r, r) = \bigcup_{\substack{a \leq r \\ i \in \{1,2\}}} A_a^i$ and $A \equiv \varphi^{SAM}(W, q)$. Then there is $w \in A$ such that $w \notin A'$. Since $|A| = |A'|$, there is $w' \in A'$ such that $w' \notin A$.

Claim 1: Neither $w, w' \in W \setminus M$ nor $w, w' \in M$ holds.

Proof of the Claim 1: Suppose by contradiction that either $w, w' \in W \setminus M$ or $w, w' \in M$ holds. We analyze both cases.

Case 1: Let $w, w' \in W \setminus M$. If $s_w > s_{w'}$, then since SAM is *minority fair* $w' \in A'$ implies that $w \in A'$ as well, which contradicts our assumption. If reversely, $s_w < s_{w'}$, once again since SAM is *minority fair*, $w \in A$ implies that $w' \in A$ as well, which contradicts our assumption.

Case 2: Let $w, w' \in M$. If $s_w > s_{w'}$, then since SAM is *minority fair*, then $w' \in A'$ implies that $w \in A'$ as well, which contradicts our assumption. If reversely, $s_w < s_{w'}$, once again since SAM is *minority fair*, $w \in A$ implies that $w' \in A$ as well, which contradicts our assumption. □

Thus, under Claim 1 there is a minority worker $\bar{w} \in M$ such that either (i) $\bar{w} \in A'$ and $\bar{w} \notin A$ or (ii) $\bar{w} \in A$ and $\bar{w} \notin A'$.

Observation 1: When $|M| \geq m \times q$, at each round $k \leq r$, we have $|(\bigcup_{\substack{a < k \\ i \in \{1,2\}}} A_a^i) \cup A_k^1| \geq m \times \sum_{a \leq k} q_a$. That is, at any round k, at the end of the step 1, we have enough number selected workers to satisfy the minority needs.

Next, we consider several cases,

Case 1: $|top_q(W) \cap M| = m \times q$. That is, there is just enough minority workers among top q highest scoring workers in W to satisfy the minority needs. Let $top_{m \times q}(M)$ be the set of those workers. By definition, $A' = top_q(W)$ and $A' \cap M = top_{m \times q}(M)$.

(i) Note that if there is $w' \in A' \cap M$ such that $w' \notin A \cap M$, then either we have $|A \cap M| < m \times q$ which violates the fact that the SAM rule respects minority rights, or we have $|A \cap M| = m \times q$ which implies there is $w'' \in A \cap M$ such that $w'' \notin A \cap M$, and since $A' \cap M = top_{m \times q}(M)$ we have $s_{w'} > s_{w''}$ which violates SAM being *minority fair*.

(ii) If, on the other hand, there is $w' \in A \cap M$ such that $w' \notin A' \cap M$, then $|A \cap M| > m \times q = |A' \cap M|$. That is, there is a round $k \leq r$ and step $i \in \{1, 2\}$ during which this extra minority worker was taken. Note that w' has a lower score than any worker $w'' \in top_{m \times q}(M)$. Consider the second step of round r. By Observation 1, $|(A \setminus A_r^2) \cap M| \geq m \times q$. By definition of the SAM, worker w' cannot be taken at this step. Now consider the first step of round r. Once again, under Observation 1 and the definition of SAM, either $A_r^1 = \emptyset$ and$|(A \setminus A_r^2) \cap M| \geq m \times q$, or $A_r^1 \neq \emptyset$, in which case $|(A \setminus A_r^2) \cap M| = m \times q$. If it is the second scenario then there is a worker $\bar{w} \in top_{m\times q}(M)$ such that $\bar{w} \notin A \cap M$ and the fact that $w' \in A \cap M$ together with $s_{\bar{w}} > s_{w'}$ contradicts *minority fairness*. If, on the other hand, it is the first scenario, then we consider the second step of round $r-1$. We continue in the same way. Since there are finite number of rounds and steps, there is a round k (simply consider round 1), where $A_k^1 \neq \emptyset$ and$|(\bigcup_{\substack{a<k \\ i\in\{1,2\}}} A_a^i) \cup A_k^1| = m \times \sum_{a\leq k} q_a$. Thus, we obtain contradiction. Hence, there is no $w' \in A \cap M$ that is$w' \notin A' \cap M$. **Case 2:** $|top_q(W) \cap M| < m \times q$. By definition, $A' \cap M = top_{m\times q}(M)$.

(i) Note that if there is $w' \in A' \cap M$ such that $w' \notin A \cap M$, then either we have $|A \cap M| < m \times q$ which violates the fact that the rule respects minority rights, or we have $|A \cap M| = m \times q$ which implies there is $w'' \in A \cap M$ such that $w'' \notin A \cap M$, and since $A' \cap M = top_{m\times q}(M)$ we have $s_{w'} > s_{w''}$ which violates SAM being *minority fair*.

(ii) If, on the other hand, there is $w' \in A \cap M$ such that $w' \notin A' \cap M$, then $|A \cap M| > m \times q = |A' \cap M|$. That is, there is round $k \leq r$ and step $i \in \{1, 2\}$ during which this extra minority worker was taken. Consider the second step of round r. By Observation 1, $|(A \setminus A_r^2) \cap M| \geq m \times q$. By definition of SAM, worker w' cannot be taken at this step. Note that w' has lower score than any worker $w'' \in top_{m\times q}(M)$. Now consider the first step of round r. Once again, under Observation 1 and the definition of SAM, either $A_r^1 = \emptyset$ and $|(A \setminus A_r^2) \cap M| \geq m \times q$, or $A_r^1 \neq \emptyset$, in which case $|(A \setminus A_r^2) \cap M| = m \times q$. If it is the second scenario then there is a worker $\bar{w} \in top_{m\times q}(M)$ such that $\bar{w} \notin A \cap M$ and the fact that $w' \in A \cap M$ together with $s_{\bar{w}} > s_{w'}$ contradicts *minority fairness*. If, on the other hand, it is the first scenario, then we consider second step of round $r-1$. We continue in the same way. Since there are finite number of rounds and steps, there is a round k (simply consider round 1), where $A_k^1 \neq \emptyset$ and $|(\bigcup_{\substack{a<k \\ i\in\{1,2\}}} A_a^i) \cup A_k^1| = m \times \sum_{a\leq k} q_a$. Thus, we obtain contradiction. Hence, there is no $w' \in A \cap M$ that is $w' \notin A' \cap M$.

Case 3: $|top_q(W) \cap M| > m \times q$. That is, there are more than enough minority workers among the top q highest scoring workers in W to satisfy minority needs. Let $top_q(W) \cap M$ be the set of those workers. By definition, $A' = top_q(W)$.

(i) Suppose there is a worker $w' \in A' \cap M$ such that $w' \notin A \cap M$. That is, there is a worker $w'' \in top_q(W)$ such that $w'' \notin A$. Then there is a worker $\bar{w} \notin top_q(W)$ but$\bar{w} \in A$. This contradicts the fact that SAM is *minority fair*, since $s_{w''} > s_{\bar{w}}$ and $w'' \notin A$ but $\bar{w} \in A$.

(ii) On the other hand, suppose there is $w' \in A \cap M$ such that $w' \notin A' \cap M$. Note that $w' \notin top_q(W)$. That is, there is round $k \leq r$ and step $i \in \{1, 2\}$ during which this extra minority worker was taken. Consider the second step of round r. Under Observation 1, $|(A \setminus A_r^2) \cap M| \geq m \times q$. By definition of SAM, worker w' cannot be taken at this step. Now consider the first step of round r. Once again, under Observation 1 and the definition of SAM, either $A_r^1 = \emptyset$ and $|(A \setminus A_r^2) \cap M| \geq m \times q$, or $A_r^1 \neq \emptyset$, in which case $|(A \setminus A_r^2) \cap M| = m \times q$. If it is the second scenario then there is a worker $\bar{w} \in top_{m \times q}(M)$ such that $\bar{w} \notin A \cap M$ and the fact that $w' \in A \cap M$ together with $s_{\bar{w}} > s_{w'}$ contradicts *minority fairness*. If, on the other hand, it is the first scenario, then we consider second step of round $r-1$. We continue in the same way. Since there are finite number of rounds and steps, there is a round k (simply consider round 1), where $A_k^1 \neq \emptyset$ and $|(\bigcup_{\substack{a<k \\ i \in \{1,2\}}} A_a^i) \cup A_k^1| = m \times \sum_{a \leq k} q_a$. Thus, we obtain contradiction. Hence, there is no $w' \in A \cap M$ that is $w' \notin A' \cap M$.

Theorem 2. The single priority rule satisfying common top, aggregation independence and permutation independence is very easy to see. Now suppose that there is a set of workers W and a rule φ^* that satisfies common top and permutation independence. Denote by w^* the common top worker. We first show by induction in the number of hires that there is an ordering of workers $(w^*, w^2, w^3, \ldots)$ such that for any sequence of single hires, that is, the hiring of one worker at a time, φ^* will consist of a single priority with that ordering. The induction base comes from the property of common top: For any institution i, if $q_I^r = ((i,1))$ then $\varphi_i^*(W, q_I^r) = w^*$. Next, by induction assumption, for any sequence of $k-1$ hires, in the first round the institution hires w^*, in the second round the next (possibly the same) institution hires w^2, etc. Now, for the sake of contradiction, assume that the rule is not a single priority rule, and without loss of generality, that the kth round is the first in which the rule does not hire following a single priority. That is, if the kth hiring is made by institution i the worker hired will be w, and if made by i', the worker will be $w' \neq w$.

By induction assumption, in the sequence of hires $((i_1,1), (i_2,1), \ldots, (i_{k-2},1), (\{i,i'\},1))$, the set of workers hired is independent of whether the last hire was made by the last institution i or i' . By permutation independence, the set of workers hired after sequences $((i_1,1), (i_2,1), \ldots, (i_{k-2},1), (i,1), (i',1))$ is the same that is hired under the permutation $((i_1,1), (i_2,1), \ldots, (i_{k-2},1), (i',1), (i,1))$. By contradiction assumption, we assumed that the last hire in the first case was w' and in the second w. Since the overall set of workers hired remains the same, it must be that the set of workers hired after the sequence $((i_1,1), (i_2,1), \ldots, (i_{k-2},1), (i,1))$ contains w and not w', and after the sequence $((i_1,1), (i_2,1), \ldots, (i_{k-2},1), (i',1))$ it contains w' and not w. This is a contradiction with the induction assumption.

We have shown, therefore, that when considering single hirings, any rule that satisfies common top and permutation independence is single priority. Finally, consider any sequence of hires $((i_1,q_1), (i_2,q_2), \ldots, (i_k,q_k))$. Since the rule is aggregation independent, the q_1 workers hired in the first round are the same as those hired in q_1 rounds, one by one:

$$\varphi^*_{i_1}\left(W,\left(\left(i_1,q_1\right)\right)\right)=\varphi^*_{i_1}\left(W,\left(\underbrace{\left(i_1,1\right),\left(i_1,1\right),\ldots,\left(i_1,1\right)}_{q_1\text{ times}}\right)\right)$$

Similarly, aggregation independence implies that all subsequent hires can be split into sequences of single hires, without changing the set of workers hired. Therefore, any rule satisfying common top, aggregation independence and permutation independence is single priority.

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