WZB

Wissenschaftszentrum Berlin
für Sozialforschung

Tommy Andersson
Lars Ehlers
Lars-Gunnar Svensson

Transferring Ownership of Public Housing to Existing Tenants: A Mechanism Design Approach

Tommy Andersson, Lars Ehlers, Lars-Gunnar Svensson
Transferring Ownership of Public Housing to Existing Tenants:
A Mechanism Design Approach

Affiliation of the authors:

Tommy Andersson
Lund University Sweden

Lars Ehlers
University of Montreal

Lars-Gunnar Svensson
Lund University Sweden

Abstract

**Transferring Ownership of Public Housing to Existing Tenants:
A Mechanism Design Approach**

by Tommy Andersson, Lars Ehlers and Lars-Gunnar Svensson[*]

This paper explores situations where tenants in public houses, in a specific neighborhood, are given the legislated right to buy the houses they live in or can choose to remain in their houses and pay the regulated rent. This type of legislation has been passed in many European countries in the last 30–35 years (the U.K. Housing Act 1980 is a leading example). The main objective with this type of legislation is to transfer the ownership of the houses from the public authority to the tenants. To achieve this goal, selling prices of the public houses are typically heavily subsidized. The legislating body then faces a trade-off between achieving the goals of the legislation and allocating the houses efficiently. This paper investigates this specific tradeoff and identifies an allocation rule that is individually rational, equilibrium selecting, and group non-manipulable in a restricted preference domain that contains "almost all" preference profiles. In this restricted domain, the identified rule is the equilibrium selecting rule that transfers the maximum number of ownerships from the public authority to the tenants. This rule is preferred to the current U.K. system by both the existing tenants and the public authority. Finally, a dynamic process for finding the outcome of the identified rule, in a finite number of steps, is provided.

Keywords: Public housing; existing tenants; equilibrium; minimum equilibrium prices; maximum trade; group non-manipulability; dynamic price process

JEL classification: C71; C78; D71; D78

---

[*] E-mail: tommy.andersson@nek.lu.se, lars.ehlers@umontreal.ca, gunnar.svensson@nek.lu.se.

1 Introduction

Public housing is a common form of housing tenure in which the property is owned by a local or central government authority. This type of tenure has traditionally referred to a situation where the central authority lets the right to occupy the units to tenants. In the last 30–35 years, however, many European countries have experienced important changes in their policies. In United Kingdom, for example, the "Right to Buy" was implemented in the U.K. Housing Act 1980. As indicated by its name, the legislation gave existing tenants the right to buy the houses they were living in (a tenant could, however, choose to remain in the house and pay the regulated rent).[1] The U.K. Secretary of State for the Environment in 1979, Michael Heseltine, stated that the main motivation behind the Act was:

> "... to give people what they wanted, and to reverse the trend of ever increasing dominance of the state over the life of the individual."

> (from "Housing Bill – Provisions and Enactment" in Keesing's Contemporary Archives v.27, January, 1981, p.30644.)

In other words, the main objective of the Act was to transfer the ownership of the houses from the local or central authority to the existing tenants. To cope with this objective, the Act also specified that tenants could buy the houses at prices significantly below the market price and that resale is not allowed within five years after purchasing the houses. In the latest update of the Act from 2012, this "discount" was specified to a maximum of £75000 or 60 percent of the house value (70 percent for an apartment) depending on which is lower.[2] As can be expected, the result of the Act was that the proportion of public housing in United Kingdom fell from 31 percent in 1979 to 17 percent in 2010.[3] Similar legislations, with similar effects, have been passed in several European countries, including Germany, Ireland, and Sweden, among others.

An important feature of the above type of legislation is that existing tenants always have the option to continue renting the houses at the regulated rents ("Right to Stay and Rent"). This option highlights an important trade-off between allocative efficiency and achieving the objectives of the legislation when existing tenants cannot afford to buy the houses at market prices (which is the typical case, and the reason for discounting the sales prices of the houses in the first place). More precisely, if the discount is "large", there will be allocative inefficiencies when there are buyers who are willing to buy the houses at market prices. On the other hand, if the prices of the houses are not discounted at all (i.e., if market prices prevail), the existing tenants prefer to continue renting, and, in this case,

---

[1]In the United Kingdom, local authorities have always had legal possibilities to sell public houses to tenants, but until the early 1970s such sales were extremely rare.

[2]"Reforming the Right to Buy in 2012 and 2013", Commons Library Standard Note, U.K. Parliament (retrieved June 12, 2013).

[3]"Housing Europe Review 2012 – The nuts and bolts of European social housing systems", European Federation of Public, Cooperative and Social Housing (Brussels).

the legislation will not achieve what it was designed to do. This trade-off is the starting point of this paper.

The formal model considered in this paper has its most direct application to the above type of housing market,[4] and its main practical implication is an extension of the U.K. Housing Act 1980 and its European equivalents. This extension generates a more efficient allocation of houses among the tenants without compromising with the main aim of the legislation. The formal model considers a pre-specified geographic area, called a neighborhood, and all existing tenants in the neighborhood are, by the public authority, asked to report their preferences over pairs of houses and prices. Given these reports, an allocation rule, which can be interpreted as a direct auction mechanism, identifies equilibrium prices for the "internal market" which consists only of the set of neighboring houses and their tenants, given the prerequisite that a tenant can buy any house in the neighborhood, including the one that he currently is possessing, or continue renting the house he currently is occupying. In the model, the public authority defines a fixed lower bound for the equilibrium prices, i.e., reservation prices, and in case an existing tenant buys the particular house he currently is living in, the tenant pays only the reservation price. One can think of these reservation prices as exogenously given and specified in the law as explained in the above. Note also that the reservation price can be interpreted as a "personalized" price for the existing tenant as the price for all other tenants of that specific house is given by the equilibrium price which is endogenously determined by the reports of all agents. We also remark that any agent have two "outside options" in the considered model, namely, the "Right to Stay and Rent" *and* the "Right to Buy" the house that they currently occupy at its reservation price. In summary, the significant difference between the U.K. Housing Act 1980, and its European equivalents, and the model considered in this paper is that we extend the situation from the case where *only* an existing tenant can buy the house that he currently is occupying to a situation where houses can be reallocated among *all* existing tenants in a neighborhood.

A few remarks about the interpretation of the model and its relation to the U.K. Housing Act 1980, and similar legislations, are in order here. First, as a tenant in our framework always can choose to continue renting the house they live in or to buy it at the reservation price, exactly as in the prevailing U.K. system, but have the opportunity to buy some other house in the neighborhood, all tenants are weakly better off in the considered model compared to the current U.K. system. Second, because all sold houses in the current U.K. system are sold at the reservation prices, but all sold houses in the considered model are sold at prices weakly higher than the reservation prices, the public authority is weakly better off in the considered model compared to the current U.K. system.[5] Third, the extension of the current U.K. system, captured in the formal model, is realistic as "mutual exchange" between tenants already is allowed in many of the European countries with a

---

[4]There are also other interpretations, e.g., a housing market where the owner of a house and the person living in the house coincides, and where a local or central government authority imposes taxes on trade of the houses.

[5]This argument is, of course, only valid if the houses that are sold in the current U.K. system also are sold in the considered model. This is always the case as later illustrated in Proposition 7.

legislated "Right to Buy" option, e.g., Ireland, Sweden, and United Kingdom.[6] Fourth, the considered model captures the idea that housing needs may change over time, e.g., a family has more children or some children move out. In such cases, a tenant may want to change house but not its housing area, and such tenants are given permission to participate in our housing market. This is not allowed in the prevailing U.K. system as resale not is allowed within five years after purchasing the house.

To solve the above described house allocation problem, it is natural to search for a price mechanism that is individually rational, equilibrium selecting, and non-manipulable. This type of rule guarantees that no tenant can loose by participating, that no further rationing mechanism is needed, and that the reported information is reliable which means that the central authority can be certain that the houses are allocated to the tenants in the neighborhood who value them most. Given the interest in these three specific axioms, the perhaps most natural candidate for an allocation rule is based on a "minimum equilibrium price vector" as this type of rule previously has been demonstrated to satisfy these specific axioms in a variety of different contexts, including, e.g., single-item auction environments (Vickrey, 1961), assignment markets (Demange and Gale, 1985), and housing markets with rent control (Andersson and Svensson, 2014). Another natural price mechanism for the above described housing market is an individually rational, equilibrium selecting, and "trade maximizing" rule. The latter axiom guarantees that the maximum number of houses are transferred to the tenants in equilibrium in perfect accordance with the U.K. Housing Act 1980 and its European equivalents.

Even if one would suspect that the above two rules always recommend the same selection as lower prices intuitively should increase trade, it turns out that this is generally not the case.[7] In fact, it is not even clear what any of the above two rules recommend due to the non-uniqueness of a minimum equilibrium price vector and the non-uniqueness of an allocation that maximizes trade. The non-uniqueness property of the two rules is a direct consequence of the fact that agents can block the trade of a house through the "Right to Stay and Rent" option. The multiplicity of a minimum equilibrium price vector has the severe consequence that *any* allocation rule based on minimum equilibrium prices is manipulable in the full preference domain. Similarly, the multiplicity of an allocation that maximizes trade has the consequence that *any* allocation rule based on maximum trade is manipulable in the full preference domain. However, by considering the preference domain where no two houses are "connected by indifference"[8], it turns out that a minimum equilibrium price vector is unique for each preference profile in the domain, and that it is possible to base an individual rational, equilibrium selecting, and group non-manipulable allocation rule on

---

[6]Mutual exchange refers to a situation where two (or more) tenants in the public housing sector swap their houses when they rent their houses from a public authority. Typical requirements for a mutual exchange to take place are that none of the tenants included in the swap owes rent, is in the process of being evicted, and is moving to a home that the landlord believes is too big or small for their circumstances.

[7]These two rules make the same selection for any profile in the preference domain if and only if there are at most two houses in the neighborhood (Proposition 2).

[8]This type of domain restriction is very mild as "almost all" preference profiles are included in the reduced domain.

this unique price vector. In addition, this rule turns out to be a maximum trade rule in the restricted domain, and it, therefore, guarantees that the maximum number of houses are transferred from the public authority to the tenants. The minimum price mechanism also recommends an outcome that is weakly preferred to the current U.K. system by both the existing tenants and the public authority.

Because the minimum price mechanism satisfies several nice properties in the restricted preference domain, as explained in the above, it is natural to search for a dynamic price process that identifies the outcome of it. This is especially true if one is interested in implementing its outcome for an existing housing market. For this reason, this paper also provides a dynamic price process that always locates the minimum equilibrium price vector in a finite number of steps.

The main results of the paper are not only of interest because of their practical relevance, they also contribute to the existing matching literature in a deeper sense. More explicitly, and to the best of our knowledge, this paper is the first to provide an individually rational, equilibrium selecting, trade maximizing, and non-manipulable allocation rule for a housing market with initial endowments (initial ownership) where monetary transfers are allowed. The main results presented in this paper are non-trivial extensions of similar and previously known results because most of the previous literature either consider the case with no initial endowments and where monetary transfers are allowed, or initial endowments but where monetary transfers are not allowed. In the following, we explain in more detail how the theoretical findings of this paper contribute to the existing and related literature.

The idea of using a minimum equilibrium price vector as a key ingredient in an individually rational, equilibrium selecting, and non-manipulable allocation rule was first advocated by Vickrey (1961) in his single-unit sealed-bid second-price auction. His principle was later generalized by Demange and Gale (1985) to the case where multiple heterogenous houses are sold.[9] The main differences between this paper and Demange and Gale (1985) is that they consider a pure two-sided market in the sense that buyers cannot block sales. Hence, their model cannot handle the case with existing tenants or initial ownership, and, consequently, not the case when buyers can block the trade of a house through the option "Right to Stay and Rent" or option "Right to Buy" (his house at its reservation price). However, both the allocation rule considered in this paper and the one in Demange and Gale (1985) share the property that the outcome of the rule is "envy-free" (Foley, 1967) in the sense that no agent wishes to swap his consumption bundle with another agent once the houses are allocated and the prices are determined by the rule (whereby here an agent can only consume other houses at their minimum equilibrium prices).[10]

A model with existing tenants and prices is studied by Miyagawa (2001). In his setting, each tenant owns precisely one house and is a seller as well as a buyer, so money and houses are reallocated through a price mechanism exactly as in this paper. Unlike this paper (and, e.g., Vickrey, 1961; Demange and Gale, 1985), Miyagawa (2001) requires

---

[9]See also Demange, Gale and Sotomayor (1985).

[10]Equilibrium and no-envy has been investigated previously in a two-sided setup with indivisibilities by, e.g., Alkan, Demange, and Gale (1991), Gale (1984), Quinzii (1984), Svensson (1983), Sun and Yang (2003), and Tadenuma and Thomson (1991).

the mechanism to be "non-bossy" (Satterthwaite and Sonnenschein, 1981) which roughly means that no tenant can change the assignment for some other tenant without changing the assignment for himself. The assumption of non-bossiness has dramatic consequences on any non-manipulable allocation rule in this setting (see also Schummer, 2000). Namely, any individually rational, non-bossy, and non-manipulable allocation rule is a fixed price mechanism. This means that the outcome of the fixed price mechanism typically is not envy-free. Because we do not require non-bossiness, our allocation rule is not a fixed price mechanism and selects always an envy-free outcome.

Individually rational and non-manipulable mechanisms for housing markets with existing tenants have been considered previously in the literature when monetary transfers not are allowed. Most notably Gale's Top-Trading Cycles Mechanism (Shapley and Scarf, 1974), has been further investigated by, e.g, Ma (1994), Postlewaite and Roth (1977), Roth (1982) and Miyagawa (2002). This mechanism always selects an outcome in the core. However, the allocation rules in these papers will, in similarity with Miyagawa (2001), not generally select an envy-free outcome, neither can they handle the case with monetary transfers.

Finally, a recent and intermediate proposal is due to Andersson and Svensson (2014) where houses are allocated among a set of potential tenants on a two-sided market where prices are bounded from above by price ceilings imposed by the government or a local administration. They define an individual rational, stable and group non-manipulable allocation rule that, in its two limiting cases, selects that same outcomes as the rule in Demange and Gale (1985) and the Deferred Acceptance Algorithm (Gale and Shapley, 1962). However, even if their mechanism is based on a minimal (rationing) equilibrium price vector, their allocation rule can neither handle existing tenants (with the outside options "Right to Stay and Rent" and "Right to Buy"), nor will it generally select an envy-free outcome.

The remaining part of this paper is organized as follows. Section 2 introduces the formal model and some of basic definitions that will be used throughout the paper. The allocation rules are introduced in Section 3 where also the main existence and non-manipulability results of the paper are stated. Section 4 provides a dynamic process for identifying the outcome of the considered allocation rule in a finite number of steps. Section 5 contains some concluding remarks. All proofs are relegated to the Appendix.

2 The Model and Basic Definitions

The *agents* and the *houses* are gathered in the finite sets $A = \{1, \ldots, n\}$ and $H = \{1, \ldots, n\}$, respectively, where $n = |A|$ is a finite natural number. Note that the number of agents and houses coincide as we assume that there is an existing tenant in each house. Agent a is the existing tenant of house h if $h = a$. It is often natural to assume that there are several owners of the houses, but since our results do not require this, it is assumed that there is a single *owner* of all houses.[11] We also remark that the owner of the

---

[11]Our results extend with only a few notational modifications to the case with multiple owners.

houses is not an existing tenant in the current interpretation of the model.[12]

Each house $h \in H$ has a price $p_h \in \mathbb{R}_+$. These prices are gathered in the price vector $p \in \mathbb{R}_+^n$ which is bounded from below by the *reservation prices* $\underline{p} \in \mathbb{R}_+^n$ of the owner. The reservation price of house h is $\underline{p}_h$. The reservation prices are arbitrary but fixed and define a feasible set of prices Ω according to:

$$\Omega = \{p \in \mathbb{R}_+^n : p \geq \underline{p}\}.$$

One can think of the reservation prices as exogenously given and specified in the law as explained in the Introduction.

Agent $a \in A$ can continue renting house $h = a$ at the given fixed rent ("Right to Stay and Rent") or buy the house he currently is living in at the owner's reservation price $\underline{p}_a$ ("Right to Buy"). Agent a can also buy house $h \neq a$ at price p_h. For notational simplicity, and without loss of generality, the fixed rents will not be introduced in the formal framework. Formally, each agent $a \in A$ consumes exactly one (consumption) *bundle*, x_a, in his *consumption set* $X_a = (H \times \mathbb{R}_+) \cup \{a\}$ where:

$$x_a = \begin{cases} a & \text{if agent } a \text{ continues to rent house } h = a, \\ (a, \underline{p}_a) & \text{if agent } a \text{ buys house } h = a \text{ at price } \underline{p}_a, \\ (h, p_h) & \text{if agent } a \text{ buys house } h \text{ at price } p_h. \end{cases}$$

Note that each agent $a \in A$ has two *outside options*, i.e., the "Right to Stay and Rent", $x_a = a$, and the "Right to Buy" the house that they currently occupy at its reservation price, $x_a = (a, \underline{p}_a)$. For convenience, we often denote the outside option of agent a simply by house a, i.e., $x_a = a$ will mean that either agent a continues renting *or* buys the house he currently is living in at its reservation price. The agent's choice between these two options does not affect other agents. For technical reasons, an agent a can also buy his own house at the price p_a, but that will not be the choice of a utility maximizing agent if $p_a > \underline{p}_a$. For simplicity, a bundle of type (h, p_h) will often be written as (h, p), i.e., $(h, p) \equiv (h, p_h)$. It is then understood that (h, p) means house $h \in H$ with price p_h at the price vector p.

Each agent $a \in A$ has *preferences* over bundles. These preferences are denoted by R_a and are represented by a complete preorder on X_a. The strict and indifference relations are denoted by P_a and I_a, respectively. Preferences are assumed to be *continuous, strictly monotonic*, and *boundedly desirable*. Strict monotonicity means that agents strictly prefer a lower price to a higher price on any given house, i.e., $(h, p_h)P_a(h, p'_h)$ if $p_h < p'_h$ for any agent $a \in A$ and any house $h \in H$. Bounded desirability means that if the price of a house is "sufficiently high", the agents will strictly prefer to keep the house they currently are living in rather than buying some other house, i.e., $aP_a(h, p_h)$ for each agent $a \in A$ and for each house $h \in H$ for p_h "sufficiently high". All preference relations R_a satisfying the above properties for agent $a \in A$ are gathered in the set $\mathcal{R}_a$. A (preference) *profile* is a list $R = (R_1, \ldots, R_n)$ of the agents' preferences. This list belongs to the set $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_n$. We also adopt the notational convention of writing a profile $R \in \mathcal{R}$ as $R = (R_C, R_{-C})$ for some nonempty subset $C \subseteq A$.

---

[12]It is, however, possible to allow for this case, see footnote 4.

A *state* is a triple (μ, ν, p), where $\mu : A \to H$ is a mapping assigning houses to the agents, $\nu : A \to \{0, 1\}$ is an assignment indicating if an agent $a \in A$ is renting, $\nu_a = 0$, or buying, $\nu_a = 1$, and $p \in \mathbb{R}^n$ is a price vector. If agent $a \in A$ is assigned house $\mu_a \in H$ and $\nu_a = 1$, he pays:

$$p_{\mu_a} = \begin{cases} p_{\mu_a} & \text{if } \mu_a \neq a, \\ \underline{p}_a & \text{if } \mu_a = a. \end{cases}$$

The assignment function μ is a bijection with the restriction $\mu_a = a$ if $\nu_a = 0$. This means that an agent cannot rent a house that currently he is not living in. Agent a is also the only agent that can buy house $h = a$ at the reservation price. We will use the simplified notation $x = (\mu, \nu, p)$ for a state, where $x \in \times_{a \in A} X_a$ and $x_a = (\mu_a, p)$ if $\nu_a = 1$ and $\mu_a \neq a$, and $x_a = a$ if $\mu_a = a$ and $\nu_a \in \{0, 1\}$. Here, it is understood that $\nu_a = 0$ only if $a P_a(a, \underline{p}_a)$.

The *cardinality* $|\nu|$ of the assignment ν indicates the number of agents who are buying a house at state (μ, ν, p), and it is defined by $|\nu| = \Sigma_{a \in A} \nu_a$.

Definition 1. For a given profile $R \in \mathcal{R}$, a price vector $p \in \Omega$ is an *equilibrium price vector* if there is a state $x = (\mu, \nu, p)$ such that the following holds for all agents $a \in A$: (i) $x_a R_a a$, and (ii) $x_a R_a(h, p)$ for all $h \in H$. If, in addition, the cardinality of the assignment ν is maximal among all states with price vector p satisfying (i) and (ii), the state x is an *equilibrium state*.

A state is therefore an equilibrium state if any agent weakly prefers his bundle to the house that he currently is renting (*individual rationality*) and to all other other houses at the given prices, and, if, in addition, trade is maximal. The latter condition reflects the fact that the owner of the houses prefers to sell the houses rather than keeping existing tenants (recall from the Introduction that this is the main goal of the U.K. Housing Act 1980 and similar legislations). We also remark that any equilibrium state is *envy-free* (Foley, 1967) in the sense that each agent weakly prefers his own bundle to the bundles assigned to the other agents.[13]

For a given profile $R \in \mathcal{R}$, the set of equilibrium prices is denoted by Π_R, and the set of equilibrium states is denoted by $\mathcal{E}_R$. Hence:

$$\Pi_R = \{p \in \mathbb{R}^n : (\mu, \nu, p) \in \mathcal{E}_R \text{ for some assignments } \mu \text{ and } \nu\}.$$

An equilibrium price vector p' is a *minimum* equilibrium price vector, at a given profile $R \in \mathcal{R}$, if (i) $p' \in \Pi_R$ and (ii) $p \leq p'$ and $p \in \Pi_R$ imply $p = p'$.

Finally, a state (μ, ν, p) is a *maximum trade equilibrium*, at a give profile $R \in \mathcal{R}$, if $(\mu, \nu, p) \in \mathcal{E}_R$ and $|\nu| \geq |\nu'|$ for all $(\mu', \nu', p') \in \mathcal{E}_R$.

3 Manipulability and Non-Manipulability Results

As already explained in the Introduction, it is natural to let a minimum equilibrium price vector be a key ingredient in an allocation rule for the considered house allocation problem

---

[13]Formally, if $(h, p) P_a x_a$ for some agent $a \in A$ at state $x = (\mu, \nu, p)$, agent a is said to *envy* the agent $j \in A$ with $\mu_j = h$. If no agent $a \in A$ envies any agent $a' \in A$ at state $x = (\mu, \nu, p)$, the state x is *envy-free*.

as this type of rule previously has been demonstrated to be individually rational, equilibrium selecting, and non-manipulable in various economic environments. Another natural price mechanism is based on individual rationality, equilibrium selection, and maximal trade. Here, the latter axiom guarantees that the rule maximizes the number of traded houses, i.e., that it achieves the objectives of the U.K. Housing Act 1980 and similar legislations. These two rules are defined formally next.

Consider a preference domain $\mathcal{R}^* \subseteq \mathcal{R}$, and let $\mathcal{E}^* = \bigcup_{R \in \mathcal{R}^*} \mathcal{E}_R$. A *rule* is a function $f : \mathcal{R}^* \to \mathcal{E}^*$ where, for each profile $R \in \mathcal{R}^*$, the rule f selects an equilibrium state $(\mu, \nu, p) \in \mathcal{E}_R$. A rule is called a *minimum price mechanism* if it, for each profile $R \in \mathcal{R}^*$, selects an equilibrium state $(\mu, \nu, p) \in \mathcal{E}_R$ where p is a minimum equilibrium price vector. A rule is called a *maximum trade rule* if it, for each profile $R \in \mathcal{R}^*$, selects an equilibrium state $(\mu, \nu, p) \in \mathcal{E}_R$ that, in addition, is a maximum trade equilibrium.

To be certain that the above two rules are well-defined, we need to establish that the equilibrium set $\mathcal{E}_R$ is nonempty for all profiles in $\mathcal{R}$ because in this case, Π_R will also be nonempty for all profiles in $\mathcal{R}$ by definition. The existence of a minimum equilibrium price vector then follows directly as the set of equilibrium prices is bounded from below by $\underline{p}$ and closed since preferences are continuous, and the existence of a maximum trade equilibrium follows by the non-emptiness of the equilibrium set.

Proposition 1. The set of equilibria $\mathcal{E}_R$ is nonempty for each profile $R \in \mathcal{R}$ and for any vector of reservation prices $\underline{p}$.

We will next, in a series of examples and propositions, investigate the similarities and differences between the above defined rules. The first example demonstrates that the selection of a minimum price mechanism and the selection of a maximum trade rule need not be identical in the full preference domain $\mathcal{R}$. This result is a bit counterintuitive as one would suspect that these rules always recommend the same selection as lower prices intuitively should increase trade.

Example 1. Let $A = \{1, 2, 3\}$ and $H = \{1, 2, 3\}$ be the sets of agents and houses, respectively, where agent $a \in A$ is the existing tenant of house $h = a$. Let $\underline{p} = (0, 0, 0)$. For each agent $a \in A$, preferences over bundles (h, p) are represented by a quasi-linear utility function $u_{ah}(p) = v_{ah} - p_h$ where the values v_{ah} are represented by real numbers. Let the utility for agent $a \in A$ of renting house $h = a$ formally be represented by v_{a0}, and:

$$V = \begin{pmatrix} v_{10} & v_{11} & v_{12} & v_{13} \\ v_{20} & v_{21} & v_{22} & v_{23} \\ v_{30} & v_{31} & v_{32} & v_{33} \end{pmatrix} = \begin{pmatrix} -2 & -2 & 3 & 2 \\ -2 & 1 & 1 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

In this case, $p = (0, 0, 0)$ is the unique minimum equilibrium price vector as $x = (\mu, \nu, p)$ is an equilibrium state for $\mu = (2, 1, 3)$ and $\nu = (1, 1, 0)$. Note next that x is the only possible selection of a minimum price mechanism and, moreover, agent 3 stays and rents his house (which he strictly prefers to buying house 3 at its reservation price). The state $x' = (\mu', \nu', p')$ is a possible selection of a maximum trade rule for $\mu' = (3, 2, 1)$, $\nu' = (1, 1, 1)$,

and $p' = (0, 1, 0)$. Note that in x', agent 2 buys house 2 for its reservation price (and not for price $p'_2 = 1$). Somewhat surprisingly, agent 3 continues renting in the minimum price equilibrium but buys house 1 in the maximum trade equilibrium. Hence, $|\nu'| > |\nu|$ which demonstrates that a minimum price mechanism and a maximum trade rule need not make identical selections. Furthermore, all agents weakly prefer the minimum price equilibrium x (where agents' utilities are $(3, 1, 0)$) to the maximum trade equilibrium x' (where agents' utilities are $(2, 1, 0)$). $\qquad\square$

The observation from Example 1 is in fact more general than it appears. This is revealed in the following proposition.

Proposition 2. Let f be a minimum price mechanism with domain $\mathcal{R}$. Then f is a maximum trade rule if and only if $1 \le |A| \le 2$.

We also remark that a maximum trade rule need not be a minimum price mechanism for any $|A| \ge 1$ on the domain $\mathcal{R}$. This can intuitively be understood by considering the special case when all agents always strictly prefer to buy the house they currently are living in to renting it or to buying some other house, i.e., $(a, \underline{p}_a) P_a a$ and $(a, \underline{p}_a) P_a (h, p_h)$ for all $a \in A$, all $h \ne a$, and all $p \ge \underline{p}$. In this case, a minimum price mechanism will have maximal trade, as all agents buy the house they are living in, and the unique minimum equilibrium price vector is given by $\underline{p} = (\underline{p}_1, \dots, \underline{p}_n)$. In this special case, however, maximal trade will be the outcome of any equilibrium selecting rule for any price vector $p \ge \underline{p}$, i.e., a maximum trade rule need not necessarily recommend the same state as a minimum price mechanism as the price vector may differ between the two rules.

The following example demonstrates that neither a minimum price mechanism nor a maximum trade rule need to make a unique price selection on the full preference domain $\mathcal{R}$.

Example 2. Let $A = \{1, 2, 3, 4\}$ and $H = \{1, 2, 3, 4\}$ be the sets of agents and houses, respectively. Let $\underline{p} = (0, 0, 0, 0)$. As in Example 1, it is assumed that preferences are represented by quasi-linear utility functions where

$$
V = \begin{pmatrix} v_{10} & v_{11} & v_{12} & v_{13} & v_{14} \\ v_{20} & v_{21} & v_{22} & v_{23} & v_{24} \\ v_{30} & v_{31} & v_{32} & v_{33} & v_{34} \\ v_{40} & v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 2 & -2 & -2 & 1 \\ 0 & -2 & 2 & -2 & 1 \end{pmatrix}.
$$

In this case, both $p' = (1, 0, 0, 0)$ and $p'' = (0, 1, 0, 0)$ are minimum equilibrium price vectors (details are provided in the proof of Proposition 3). This follows since both $x' = (\mu', \nu', p')$ and $x'' = (\mu'', \nu'', p'')$ are equilibrium states for $\mu' = (1, 3, 4, 2)$, $\nu' = (1, 1, 1, 1)$, $\mu'' = (3, 2, 1, 4)$, and $\nu'' = (1, 1, 1, 1)$. Note that at state x', agent 1 buys his house at its reservation price 0, at state x'' agent 2 buys his house at its reservation price 0, and that trade is maximized at states x' and x'' since $|\nu'| = |\nu''| = |A|$. $\qquad\square$

The multiplicity of a minimum equilibrium price vector and a maximum trade equilibrium is a direct consequence of the fact that agents can block the trade of a house through their

outside option "Right to Stay and Rent" or "Right to Buy" (at the reservation price). This has the severe consequence that *any* minimum price mechanism and *any* maximum trade rule is manipulable on the full preference domain, at least if there are more than three houses on the housing market. The following notion of group manipulability and (group) non-manipulability is employed.

Definition 2. A rule f with domain $\mathcal{R}^* \subseteq \mathcal{R}$ is *manipulable* at a profile $R \in \mathcal{R}^*$ by a nonempty group of agents $C \subseteq A$ if there is a profile $R' = (R'_C, R_{-C}) \in \mathcal{R}^*$, and two states $f(R) = x = (\mu, \nu, p)$ and $f(R'_C, R_{-C}) = x' = (\mu', \nu', p')$ such that $x'_a P_a x_a$ for all $a \in C$. If the rule f is not manipulable by any group $C \subseteq A$ at any profile $R \in \mathcal{R}^*$, it is *group non-manipulable*. If the rule f is not manipulable by any group of size one (or any agent) at any profile $R \in \mathcal{R}^*$, it is *non-manipulable*.

Proposition 3. A minimum price mechanism f is non-manipulable on the domain $\mathcal{R}$ if and only if $1 \leq |A| \leq 3$.

The result in Proposition 3 does not generally carry over to maximum trade rules. It is indeed true that any maximum trade rule can be manipulated if $|A| > 3$ (see Proposition 4) but due to the large number of possible maximum trade rules, it depends on the specific maximum trade rule if it is manipulable or not when $1 \leq |A| \leq 3$. For example, as a minimum price mechanism is a maximum trade rule in the interval $1 \leq |A| \leq |2|$, by Proposition 2, it follows from Proposition 3 that a maximum trade rule that, in addition, is a minimum price mechanism is non-manipulable if $1 \leq |A| \leq 2$. On the other hand, a maximum trade rule that avoids to make the same selection as a minimum price mechanism for any profile in $\mathcal{R}$ whenever possible is non-manipulable on the domain $\mathcal{R}$ if and only if $|A| = 1$.[14]

Proposition 4. A maximum trade rule f is manipulable on the domain $\mathcal{R}$ if $|A| > 3$.

The implication from Propositions 3 and 4 is that if one searches for non-manipulable rules on the full preference domain, one cannot search in the class of minimum price mechanisms or the class of maximum trade rules, at least, if one is interested in housing markets containing more than three houses. Of course, there are non-manipulable rules also on the full preference domain, e.g., a rule which, for any profile, sets prices "sufficiently high" so that all agents prefer either renting or buying the house they currently occupy. As this rule always recommends an identical outcome as the rule which currently is used in United Kingdom, it will not be of any interest to a public authority that aims to transfer *more* houses to tenants compared to the existing U.K. system.

---

[14]The interested reader may consult the following example which is the key in a formal proof. Suppose, as in Example 1, that preferences are represented by quasi-linear utility functions, and that $A = H = \{1, 2\}$, $v_{10} = v_{11} = v_{20} = -2$, $v_{12} = 0$, $v_{21} = 1$, and $v_{22} = 0$. In this case, a maximum trade rule that do not make the same selection as a minimum price mechanism (whenever possible) will select the state $x = (\mu, \nu, p)$ where $\mu = (2, 1)$, $\nu = (1, 1)$, and $p = (\alpha, 0)$ for some $0 < \alpha \leq 1$. However, if agent 2 misrepresents and instead reports $\tilde{v}_{21} = 0$, agent 2 will still be assigned house 1 but at price 0. Hence, the gain by misrepresenting is $\alpha > 0$.

In the following, we will demonstrate that by excluding some profiles from the domain $\mathcal{R}$ and instead consider the reduced preference domain $\tilde{\mathcal{R}} \subset \mathcal{R}$ where no two outside options, or (using our convenience) no two houses, are "connected by indifference" at any price vector $p \in \Omega$, a minimum equilibrium price vector is unique for all profiles in the reduced domain, and that this price vector is a key ingredient for an individually rational, equilibrium selecting, and group non-manipulable rule, which, in addition, maximizes trade.

Definition 3. For a given profile $R \in \mathcal{R}$, two houses, h and h', in H are *connected by indifference* if there is a price vector $p \in \Omega$, a sequence of distinct agents $(a_1, \ldots, a_q)$, and a sequence of distinct houses $(h_1, \ldots, h_{q+1})$ for $q \geq 1$ such that:

(i) $h = h_1 = a_1$, and $h' = h_{q+1} = a_q$,

(ii) $a_1 I_{a_1}(h_2, p)$, and $a_q I_{a_q}(h_q, p)$,

(iii) $(h_j, p) I_{a_j}(h_{j+1}, p)$ for $2 \leq j \leq q - 1$ if $q > 2$.

The subset of $\mathcal{R}$ where no two houses are connected by indifference is denoted by $\tilde{\mathcal{R}}$.

In Definition 3, house a_1 stands for agent a_1's outside option of continuing renting a_1 or buying a_1 at its reservation price (and similarly, for house a_q and agent a_q). We remark that the above type of domain restriction contains "almost all" preference profiles.[15] In fact, one can argue that the restriction of profiles to $\tilde{\mathcal{R}} \subset \mathcal{R}$ is an assumption of the same character as assuming strict preferences in the absence of monetary transfers, as, e.g., Gale and Shapley (1962), Shapley and Scarf (1974), Roth (1982) and Ma (1994) among others, as also this assumption is mild if preferences are chosen randomly since the probability of an indifference then is zero. The following example illustrates a natural preference domain satisfying Definition 3.

Example 3. Consider the domain $\dot{\mathcal{R}} \subset \mathcal{R}$ where "indifference with the endowment" is excluded, i.e., $R \in \dot{\mathcal{R}}$ if and only if either $a P_a(h, p_h)$ or $(h, p_h) P_a a$ for all $a \in A$ and all (h, p_h).[16] Exclusion of indifference with the endowment arises naturally when agents have a "status-quo bias" meaning that once an agent is indifferent between keeping the house he currently is living or buying some other house, the agent keeps his original house. Assumptions with a similar meaning has previously been made by, e.g., Bogomolnaia and Moulin (2004) in dichotomous matching. $\qquad\square$

---

[15]Informally, this can be understood by considering any profile $R \in \mathcal{R}$, and any sequence of distinct agents $(a_1, \ldots, a_q)$ and any sequence of distinct houses $(h_1, \ldots, h_{q+1})$ with $h_1 = a_1$ and $h_{q+1} = a_q$. Suppose now that $a_1 I_{a_1}(h_2, p)$ and $(h_j, p) I_{a_j}(h_{j+1}, p)$ for $2 \leq j \leq q - 1$. In this case, p_{h_q} will be uniquely determined by continuity and monotonicity of the preferences. However, if preferences are chosen randomly, the probability is zero that $a_q I_{a_q}(h_q, p)$, i.e., that houses h and h' are connected by indifference. Hence, very few profiles are excluded in $\tilde{\mathcal{R}}$ compared to $\mathcal{R}$. See Andersson and Svensson (2014) for a thorough analysis of a similar domain restriction.

[16]For technical purposes, we also need to assume that the set $\{p_h \in \mathbb{R} : (h, p_h) R_h a\}$ is closed for all $h \in H$. This assumption guarantees that for a decreasing convergent equilibrium price sequence, the assignment converges as well.

The next example demonstrates how to design reservation prices such that no two houses are connected by indifference given that preferences are represented by quasi-linear utility functions where the valuations are rational numbers.

Example 4. Let $\mathbb{Q}$ denote the set of all rational numbers. For any agent $a \in A$, let $\mathcal{R}_a^q$ consist of all quasi-linear utility functions where $v_{ah} \in \mathbb{Q}$ for all $h \in H \cup \{0\}$. Now choose reservation prices $\underline{p}$ such that $\underline{p}_h \in \mathbb{R}_+ \backslash \mathbb{Q}$ for all $h \in H$, and $\underline{p}_h - \underline{p}_{h'} \in \mathbb{R} \backslash \mathbb{Q}$ for any distinct houses $h, h' \in H$. Then it is easy to verify that $\mathcal{R}_1^q \times \cdots \times \mathcal{R}_n^q \subset \tilde{\mathcal{R}}$.

Note that all our results apply to any subdomain $\mathcal{R}^* \subseteq \tilde{\mathcal{R}}$, i.e., for applying our results it suffices to construct/design such subdomains and by defining the rules on this restricted subdomain.

The first main result of the paper demonstrates that the set of equilibrium prices Π_R has a unique minimum equilibrium price vector at any profile $R \in \tilde{\mathcal{R}}$. This result also explains why there are two minimum equilibrium price vectors in Example 2. More precisely, houses $h = 1$ and $h' = 3$ are connected by indifference at prices $p' = (1, 0, 0, 0)$. To see this, consider agents 1 and 2 and the sequence of houses $(1, 3, 2)$. In this case, $q = 2$, $a_1 = 1$, $a_2 = a_q = 2$, $1I_1(3, 0)$, and $2I_2(3, 0)$ so all conditions of Definition 3 are satisfied (note that the last condition in the definition is irrelevant because $q = 2$), i.e., houses $h = 1$ and $h' = 3$ are connected by indifference at prices $p' = (1, 0, 0, 0)$.

Theorem 1. There is a unique minimum equilibrium price vector $p^* \in \Pi_R$ for each profile $R \in \tilde{\mathcal{R}}$.

From Theorem 1, it is clear that the non-uniqueness problem, previously illustrated in Example 2, disappears on the restricted domain $\tilde{\mathcal{R}}$. What may not be so obvious, in the light of Proposition 2, is that a minimum price mechanism always is a maximum trade rule on the restricted domain $\tilde{\mathcal{R}}$.

Proposition 5. Let f be a minimum price mechanism with domain $\tilde{\mathcal{R}}$. Then f is a maximum trade rule.

The last result of this section demonstrates that a minimum price mechanism, defined on the domain $\tilde{\mathcal{R}} \subset \mathcal{R}$, is group non-manipulable for all profiles in $\tilde{\mathcal{R}}$. Note also that this mechanism always selects an individual rational equilibrium outcome that maximizes trade at each profile in $\tilde{\mathcal{R}}$ by the definition of the rule and Proposition 5.

Theorem 2. Let f be a minimum price mechanism with domain $\tilde{\mathcal{R}}$. Then f is group non-manipulable.

4   A Dynamic Price Process

This section provides a dynamic process for identifying a minimum equilibrium price vector in a finite number of steps. As an arbitrary but fixed profile $R \in \mathcal{R}$ is considered throughout

the section, we will for notational simplicity drop the profile notation R in the equilibrium sets $\mathcal{E}_R$ and Π_R, and instead write $\mathcal{E}$ and Π, respectively.

The key in the dynamic process is a sequence of minimum equilibrium price vectors $(p^1, \ldots, p^T)$ consistent with a sequence of fixed assignments $(\nu^1, \ldots, \nu^T)$. This type of sequence is called a "Dutch price sequence", and to define it in more detail, some additional notation needs to be introduced. For this purpose, consider a fixed assignment ν (i.e., an assignment where, for each agent, is it exogenously given whether or not they rent a house), and let Π^ν be the set of equilibrium prices consistent with the fixed assignment ν. Such a set of equilibrium prices is called a price regime. Formally, for any fixed assignment ν, a set $\Pi^\nu \subset \Pi$ is defined to be a *price regime* if:

$$\Pi^\nu = \{p \in \Omega : \text{ there is an assignment } \mu \text{ such that } p \in \Pi \text{ for } x = (\mu, \nu, p)\}.$$

For each assignment ν, the price regime Π^ν is a semi-lattice that is closed and bounded below and, hence, has a unique minimum equilibrium price vector $p^{\nu*} \in \Pi^\nu$. Let $x = (\mu, \nu, p) \in \mathcal{E}$ be an equilibrium state and define a price function $\pi : \mathcal{E} \to \Omega$ according to $\pi(x) = p^{\nu*}$. Hence, $\pi(x)$ is the unique minimum equilibrium price in the price regime defined by the assignment ν at the state x. Further, define a correspondence ξ from minimum price vectors $p^{\nu*}$ to equilibrium states according to:

$$\xi(p^{\nu*}) = \{x' \in \mathcal{E} : x' = (\mu', \nu', p^{\nu*}) \text{ for some assignments } \mu' \text{ and } \nu'\}.$$

Definition 4. A (possibly infinite) price sequence $(p^t)_{t=1}^T$ of equilibrium prices is a *Dutch price sequence* if:

(i) $p^1 = \pi(x^0)$ for some equilibrium state $x^0 = (\mu^0, \nu^0, p^0)$ with $\mu_a^0 = a$ for all $a \in A$,

(ii) there is a supporting sequence $(x^t)_{t=1}^T$ of equilibrium states such that for some $x^t \in \xi(p^t)$, it holds that $p^{t+1} = \pi(x^t)$,

(iii) $p^T = \pi(x^{T-1})$ and for each $x \in \xi(p^T)$ and $p = \pi(x)$, it holds that $p = p^T$ whenever the price sequence $(p^t)_{t=1}^T$ ends at a finite step T.

To get a better understanding of a Dutch price sequence, consider Figure 1 and the price vector p^0 where all agents prefer to keep the house they currently are living in by either renting or buying it. Such a price vector always exists as this always will be the case for "sufficiently high" prices. But then there is an equilibrium state $x^0 = (\mu^0, \nu^0, p^0)$ as defined in Definition 4(i). Recall next that for the price vector p^0, there is a corresponding price regime Π^{ν^0} with a unique minimum equilibrium price vector. This vector is denoted by $p^1 = \pi(x^0)$. Now, given prices p^1, there is a corresponding supporting equilibrium state $x^1 = (\mu^1, \nu^1, p^1)$. Note that this state is not necessarily unique. In the left panel of Figure 1, for example, there are two supporting equilibrium states, $\hat{x}^1 = (\ldots, \hat{x}_i^1, \hat{x}_j^1, \ldots)$ and $\tilde{x}^1 = (\ldots, \tilde{x}_i^1, \tilde{x}_j^1, \ldots)$, with corresponding minimum equilibrium price vectors $\hat{p}^2 = \pi(\hat{x}^1)$ and $\tilde{p}^2 = \pi(\tilde{x}^1)$. Any of these price vectors may be chosen and the arguments can be repeated in the exact same fashion. In this way, a Dutch price sequence, $(p^1, p^2, \ldots, p^t, \ldots)$,
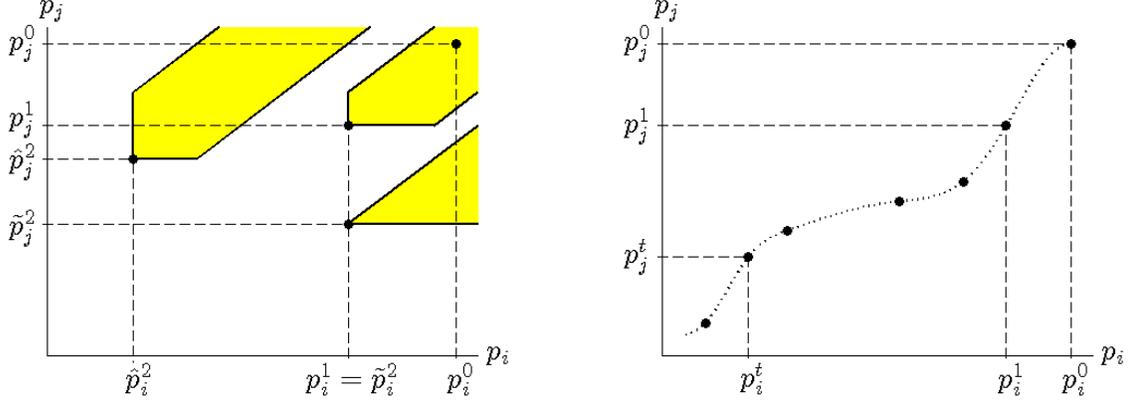
Figure 1: The left panel of the figure illustrates that a Dutch price sequence need not be unique for a given profile in $\mathcal{R}$, and the right panel of the figure illustrates a Dutch price sequence $(p^1, p^2, \ldots, p^t, \ldots)$.

as illustrated in the right panel of Figure 1, will be obtained (disregard the dotted path that connects the price vector in the Dutch price sequence for the moment, it will be explained later in this section). What may not be so obvious is that such a sequence always contains a finite number of equilibrium price vectors. This is demonstrated in the following proposition.[17]

Proposition 6. For any given profile $R \in \mathcal{R}$, a Dutch price sequence $(p^t)_{t=1}^{T}$ contains only a finite number of price vectors, i.e., $T < \infty$.

We next remark that any Dutch price sequence satisfies two important properties. First, $p^{t+1} \leq p^t$ and $p^{t+1} \neq p^t$ for any t in the sequence. This follows since $p^t \in \Pi^{\nu^t}$, and p^{t+1} is the minimum price vector in the price regime Π^{ν^t}. Second, the cardinality of the assignment ν is weakly increasing in the sequence, i.e., if the number of agents that are buying houses at t and $t + 1$ are denoted by $|\nu^t|$ with $|\nu^{t+1}|$, respectively, it must be the case that $|\nu^{t+1}| \geq |\nu^t|$. This follows since, at an equilibrium state, the number of buying agents is always maximal at the given price vector. A change from the price regime Π^{ν^t} to the price regime $\Pi^{\nu^{t+1}}$ entails a change of agents that are buying, but the number of buying agents cannot decrease.

As is clear from the above description, there may be several Dutch price sequences for a given profile $\mathcal{R}$ and a given initial start prices p^0 (see, e.g., the left panel of Figure 1). However, if the profile is selected from the domain $\tilde{\mathcal{R}}$ where no two houses are "connected by indifference", there will be exactly one Dutch price sequence for the given profile. Even more strikingly, the end point of the sequence must be the, by Theorem 1, unique minimum equilibrium price vector $p^* \in \Pi$ for the given profile $R \in \tilde{\mathcal{R}}$. Both these results are formally stated in the last theorem of the paper.

Theorem 3. For any given profile $R \in \tilde{\mathcal{R}}$, a Dutch price sequence $(p^t)_{t=1}^{T}$ is unique, and p^T is the unique minimum equilibrium price vector in Π_R.

---

[17]Note also that the above arguments can be use to construct a Dutch price sequence for any profile in $\mathcal{R}$ and any vector of reservation prices. Hence, there always exists a Dutch price sequence.

A final observation is that the price vectors in the a unique Dutch price sequence can be identified and connected by applying the "Vickrey-Dutch Auction" (Mishra and Parkes, 2009) in the case when preferences are quasi-linear, and by the Dutch counterpart of the "SA Auction" in Morimoto and Serizawa (2014) in the more general case. More explicitly, for any given profile in $\tilde{\mathcal{R}}$, the unique minimum equilibrium price vector can be obtained by adopting the following dynamic price process.

The Dynamic Price Process. Initialize the price vector to "sufficiently high" prices p^0 in the sense that each agent prefers not to buy any house they currently are *not* living in, i.e., $x_a^0 = a$ for all $a \in A$. Then for each Step $t := 0, \ldots, T$:

Step t. For the given p^t and the fixed assignment ν^t, identify the unique minimum equilibrium price vector $p^{t+1} = \pi(x^t)$ in the corresponding price regime Π^{ν^t}. If $p^{t+1} = p^t$, stop. Otherwise set $t := t + 1$ and continue. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

If one adopts the Vickrey-Dutch Auction or the Dutch counterpart of the SA Auction to identify the unique minimum equilibrium price vector in the corresponding price regime Π^{ν^t}, it follows directly from the above findings and the convergence results in Mishra and Parkes (2009, Theorem 2), and Morimoto and Serizawa (2014, Proposition 5.1) that the above process is well-defined and that it will converge to the unique minimum equilibrium price vector, for the given profile $R \in \tilde{\mathcal{R}}$, in a finite number of steps. In the right panel of Figure 1, the above procedure will generate the finite path which is represented by the dotted line.

5   Concluding Remarks

In the special case of the formal model where A consists of *one* agent, the situation in the U.K. Housing Act 1980 and its European equivalents are fully reflected in our framework. That is, the single tenant is given a take-it-or-leave-it offer either to buy the house at a fixed and discounted price, or to continue renting the house. If the agent would report his preferences to the local or central administration, the minimum equilibrium price mechanism would recommend the real-world outcome. Hence, the model considered in this paper can be seen as a representation of the public housing market in United Kingdom today where each tenant is regarded as a "separate housing market". This situation will most likely cause allocative inefficiencies as the houses are not necessarily allocated to the tenants who value them most. The compromise proposed in this paper is to merge some of these "separate housing markets" into a new and larger housing market (e.g., all houses in a specific neighborhood). In this way, houses can be more efficiently allocated among tenants but tenants are, at the same time, protected against "too high" living costs.

   As a tenant always has the option to buy the particular house that he is living in at the fixed reservation price or to continue renting it at the regulated rent, in the formal framework, it is clear that all agents weakly prefer the outcome of the investigated rule to the current U.K. system. In fact, also the public authority weakly prefers the outcome of

the minimum equilibrium price mechanism to the prevailing U.K. system as the revenue from the sales always is weakly higher in the suggested rule. This result is formally stated in the last proposition of the paper.

Proposition 7. For any profile $R \in \mathcal{R}$, the minimum price mechanism generates a weakly higher revenue to the public authority compared to the current U.K. system.

We have left a number of interesting research questions for future research, and we believe that two of them stand out more than the others. First, is the domain where no two houses are "connected by indifference" at any price vector, the maximal domain under which a minimum price mechanism is non-manipulable? Second, is the minimum price mechanism the only individual rational, equilibrium selecting, trade maximizing, and non-manipulable rule on the domain $\tilde{\mathcal{R}}$? Currently, we do not have an answer to these two questions but it would be interesting to know them, especially since the results of this paper have some very relevant policy implications.

Appendix: Proofs

The Appendix contains the proofs of all results. It also contains some additional lemmas, definitions, and concepts. All results are proved in the same order as they are presented in the main text of the paper, except Proposition 3 which is proved directly after Theorem 2.

Proposition 1. The set of equilibria $\mathcal{E}_R$ is nonempty for each profile $R \in \mathcal{R}$ and for any vector of reservation prices $\underline{p}$.

Proof. Since each house is boundedly desirable for each agent $a \in A$ at each profile $R \in \mathcal{R}$, by assumption, there is a price vector $\bar{p} > \underline{p}$ such that $aP_a(h, \bar{p})$ for all $a \in A$ and all $h \in H$. But then, $x = (\mu, \nu, \bar{p})$ constitutes an equilibrium state if $\mu_a = a$ for all $a \in A$, $\nu_a = 1$ if $(a, \underline{p}_a)R_a a$, and $\nu_a = 0$ if $aP_a(a, \underline{p}_a)$. Hence, $x \in \mathcal{E}_R$, and, consequently, $\mathcal{E}_R \neq \emptyset$ for each profile $R \in \mathcal{R}$. $\square$

Proposition 2. Let f be a minimum price mechanism with domain $\mathcal{R}$. Then f is a maximum trade rule if and only if $1 \leq |A| \leq 2$.

Proof. We first prove that f is a maximum trade rule *if* $1 \leq |A| \leq 2$. For $A = \{1\}$, this follows directly as the single agent is given a take-it-or-leave-it offer to either to buy the house at price $\underline{p}_1$ or to continue renting it, and the fact that the agent will continue to rent only if $1P_1(1, \underline{p}_1)$. Suppose now that $A = \{1, 2\}$ but that f is not a maximum trade rule on the domain $\mathcal{R}$. This means that there exists a profile $R \in \mathcal{R}$ and two states, $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_R$, where the former state is selected by a minimum price mechanism and the latter by a maximum trade rule and $|\nu| < |\nu'|$. Because $|\nu| > |\nu'|$, there must be an agent $a_l \in A$ with $\mu_{a_l} = a_l$, $\nu_{a_l} = 0$, and $\nu'_{a_l} = 1$. Consequently, $\mu_{a_l} = a_l$, $\mu_{a_k} = a_k$, $\mu'_{a_l} = a_k$ and $\mu'_{a_k} = a_l$ for $l \neq k$ as $|A| = 2$. Note next that $x_{a_l} R_{a_l} x_{a_k}$ and $x_{a_k} R_{a_k} x_{a_l}$ as x is an equilibrium state. Because $\mu_{a_j} = a_j$ for $j = 1, 2$, it must be the case

17

that $p_{\mu_j} \leq p'_{\mu_j}$ for $j = 1, 2$ otherwise x' cannot be an equilibrium state. Now, $p_{\mu_j} < p'_{\mu_j}$ for some j contradicts that p is a minimum equilibrium price vector in Π_R. Hence, $p_{\mu_j} = p'_{\mu_j}$ for $j = 1, 2$, and, consequently, $x_{a_l} I_{a_l} x_{a_k}$ and $x_{a_k} I_{a_k} x_{a_l}$ as x' is an equilibrium state. But then $x'' = (\mu'', \nu'', p'') = (\mu', \nu', p)$ is an equilibrium state at the minimum equilibrium price vector p with $|\nu''| > |\nu|$ contradicting that the cardinality $|\nu|$ is maximal at state x. Hence, state x cannot be an equilibrium.

The proof that f is a maximum trade rule on the domain $\mathcal{R}$ *only if* $1 \leq |A| \leq 2$ follows directly from Example 1. To see this, note that in order to prove the result, it suffices to find a profile $R \in \mathcal{R}$ for an arbitrary $|A| \geq 3$ where the rule can be manipulated by some agent. Consider now Example 1, and suppose that we add an arbitrary but finite number of agents to the set $A = \{1, 2, 3\}$, and that $v_{j0} = 0$ and $v_{jk} = -2$ for $j = 4, \ldots, n$, and all $k \in A^*$ where $A^* = \{1, \ldots, 3, 4, \ldots, n\}$. Suppose, in addition, that $v_{jk} = -2$ for $j = 1, 2, 3$ and $k = 4, \ldots, n$. The added agents will always rent the house they currently live in and will not affect the outcome of the rule f for agents 1, 2, and 3. Because Example 1 demonstrates that a minimum price mechanism need not make an identical selection as a maximum trade rule when $|A| = 3$, the result will then carry over to the case when $|A| \geq 3$. $\qquad\square$

Proposition 4. A maximum trade rule f is manipulable on the domain $\mathcal{R}$ if $|A| > 3$.

Proof. We will prove the proposition by identifying a profile $R \in \mathcal{R}$ where some agent can manipulate the outcome of an arbitrary maximum trade rule f when $|A| > 3$. Using the same arguments as in the proof of Proposition 2, it is sufficient to demonstrate the result for $|A| = 4$. As in Example 1, let preferences be represented by quasi-linear utility functions where:

$$
V = \begin{pmatrix} v_{10} & v_{11} & v_{12} & v_{13} & v_{14} \\ v_{20} & v_{21} & v_{22} & v_{23} & v_{24} \\ v_{30} & v_{31} & v_{32} & v_{33} & v_{34} \\ v_{40} & v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 & -2 \\ 0 & 2 & -2 & -2 & 1 \\ 0 & -2 & 2 & -2 & 1 \end{pmatrix}.
$$

In the remaining part of the proof, we let the profile $R \in \mathcal{R}$ denote the preferences that are represented by the above values. Note first that $x = (\mu, \nu, p)$ is an equilibrium state for $\mu = (1, 3, 4, 2)$, $\nu = (1, 1, 1, 1)$, and $p = (1, 0, 0, 0)$. Note, in particular, that $|\nu| = 4$. This also means that any selection $x' = (\mu', \nu', p')$ of a maximum trade rule f at profile R must have the property that $|\nu'| = 4$. But then it must also be the case that $\mu'_2 = 3$ and $p'_3 = 0$ at any selection x' of a maximum trade rule. This follows because if $\mu'_2 \neq 3$, then it must be that case that $\mu'_2 = 2$ and $\nu'_2 = 0$ otherwise agent 2 will envy the agent a with $\mu'_a = 2$ at any feasible price vector. This contradicts that x' is an equilibrium. But $\mu'_2 = 2$ and $\nu'_2 = 0$ contradicts that x' is selected by a maximum trade rule at profile R as this implies that $|\nu'| < 4$. Hence, $\mu'_2 = 3$ and $p'_3 = 0$. By using similar arguments, it is clear that $\mu'_1 = 1$ and $\nu'_1 = 1$. As a consequence, $\mu'_3 = 4$, $\nu'_3 = 1$, $p'_1 - p'_4 \geq 1$, and $\underline{p}_4 = 0 \leq p'_4 \leq 1$. But if $\mu'_3 = 4$ and $0 \leq p'_4 \leq 1$ at any selection $x' = (\mu', \nu', p')$ of a maximum trade rule f at profile R, the maximal utility that agent 3 can obtain at profile R equals $v_{34} - 0 = 1$.

18

Consider next the profile $\tilde{R} \in \mathcal{R}$ where all agents except agent 3 have the same values as in profile R, and:

$$\tilde{v}_{3h} = (0, 0, -2, -2, -2),$$

i.e., agent 3 misrepresents his values. Note first that $|\nu''| \leq 3$ for all $x'' \in \mathcal{E}_{\tilde{R}}$. To see this, suppose that $|\nu''| = 4$ for some $x'' \in \mathcal{E}_{\tilde{R}}$. By using the similar arguments as in the above, it must then be the case that $\mu_2'' = 3$, $\mu_3'' = 1$, and $p_1'' = p_3'' = 0$. But then agent 1 must be assigned house 2 or 4 and will, consequently, envy agents 2 and 3 at any feasible price vector. This contradicts that x'' is an equilibrium. Hence, $|\nu''| \leq 3$ for all $x'' \in \mathcal{E}_{\tilde{R}}$.

We next remark that $|\nu''| = 3$ for any maximum trade rule f at profile $\tilde{R}$. This follows directly as $x'' = (\mu'', \nu'', p'')$ is an equilibrium state at profile $\tilde{R}$ for $\mu'' = (3, 2, 1, 4)$, $\nu'' = (1, 0, 1, 1)$, and $p'' = (0, 1, 0, 0)$.

The final part of the proof demonstrates that $\mu_3'' = 1$ and $p_3'' = 0$ for any selection x'' made by a maximum trade rule f at profile $\tilde{R}$. This completes the proof of the proposition as the utility of agent 3 when misrepresenting and when reporting truthfully then equals $u_{31}(p'') = 2 - 0 = 2$ and (at most) $v_{34} - 0 = 1$, respectively. Suppose first that $\mu_3'' \neq 1$. Then it must be the case that $\mu_3'' = 3$ and $v_3'' = 0$ because if this is not the case, then agent 3 will prefer continuing renting, i.e. $3\tilde{P}_3 x_3''$, at any feasible price vector, contradicting individual rationality of x''. But if $\mu_3'' = 3$ and $v_3'' = 0$, it must be the case that $\mu_2'' = 2$ and $\nu_2'' = 0$ because if this is not the case, then agent 2 will envy the agent a with $\mu_a'' = 2$ at any feasible price vector contradicting that x'' is an equilibrium. But if $\nu_2'' = \nu_3'' = 0$, then $|\nu''| < 3$ which contradicts that x'' is selected by a maximum trade rule f. Hence, $\mu_3'' = 1$. But if $\mu_3'' = 1$, then it must be the case that $p_3'' = 0$ because if this not is the case, then agent 3 will envy the agent a with $\mu_a'' = 3$ at any feasible price vector contradicting that x'' is an equilibrium. Hence, $\mu_3'' = 1$ and $p_3'' = 0$ as desired. $\qquad\square$

To prove the subsequent results, some consequences of the domain restriction (stated as lemmas) are derived for which some additional concepts are needed.

Let $q > 1$ and $a_j \in A$ for $1 \leq j \leq q$. Given two assignments μ and μ', a *trading cycle* from μ to μ' is a sequence $G = (a_1, \ldots, a_q)$ of distinct agents such that $\mu_{a_j}' = \mu_{a_{j+1}}$, for $1 \leq j < q$ and $\mu_{a_q}' = \mu_{a_1}$. For simplicity, we will use the same notation for the sequence G and the corresponding set $G = \{a_1, \ldots, a_q\}$. Note that the complete trade from μ to μ' can be decomposed uniquely into a number of trading cycles.

Let $R, R' \in \tilde{\mathcal{R}}$ be two profiles such that $R' = (R_C', R_{-C})$ for some $C \subseteq A$ (where possibly $C = \emptyset$ and $R = R'$), and consider two equilibrium states $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ where $x_a' P_a x_a$ for all $a \in C$. Let H_0, H_1, H_2, and H_3 be defined as:

$$
\begin{aligned}
H_0 &= \{a \in A : \mu_a' = \mu_a = a\}, \\
H_1 &= \{h \in H : p_h' < p_h\} \backslash H_0, \\
H_2 &= \{h \in H : p_h' = p_h\} \backslash H_0, \\
H_3 &= \{h \in H : p_h' > p_h\} \backslash H_0.
\end{aligned}
$$

Lemma 1. Let R and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{\mathcal{R}}$, and $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states. Let $G = (a_1, \ldots, a_q)$ be a trading cycle from μ to μ', and $(\mu_{a_1}, \ldots, \mu_{a_q})$ the corresponding set of houses.

(i) If $a_k \in G$ and $x'_{a_k} P_{a_k} x_{a_k}$, then $\mu'_{a_k} \in H_1$.

(ii) If $\mu_a \in H_1$, $\mu'_a \in H_2 \cup H_3$ and $a \notin C$, then $x_a = a$, $\mu'_a \in H_2$, and $x_a I_a x'_a$.

Proof. Part (i) of the lemma is proved by contradiction. Assume that $a_k \in G$ and $x'_{a_k} P_{a_k} x_{a_k}$, but that $\mu'_{a_k} \notin H_1$. Note first that $\mu'_{a_k} \neq a_k$. To see this, suppose that $\mu'_{a_k} = a_k$ or, equivalently, that $x'_{a_k} = a_k$. Because x is an equilibrium state, it must be the case that $x_{a_k} R_{a_k} a_k$. But then $x_{a_k} R_{a_k} x'_{a_k}$, which contradicts the assumption that $x'_{a_k} P_{a_k} x_{a_k}$. Hence, $\mu'_{a_k} \neq a_k$ and $\nu'_{a_k} = 1$. Note next that $x_{a_k} R_{a_k} (\mu'_{a_k}, p)$ and $(\mu'_{a_k}, p) R_{a_k} (\mu'_{a_k}, p')$ because x is an equilibrium state and $\mu'_{a_k} \notin H_1$, respectively. But then $x_{a_k} R_{a_k} (\mu'_{a_k}, p')$. Now, $x'_{a_k} = (\mu'_{a_k}, p')$ because $\mu'_{a_k} \neq a_k$. Consequently, $x_{a_k} R_{a_k} x'_{a_k}$ which, again, contradicts the assumption that $x'_{a_k} P_{a_k} x_{a_k}$. Hence, $\mu'_{a_k} \in H_1$.

To prove prove Part (ii) of the lemma, note that because $\mu_a \in H_1$ and $\mu'_a \in H_2 \cup H_3$, by assumption, it follows that $\mu_a \neq \mu'_a$ and that agent a must belong to some trading cycle from μ to μ'. Since $a \notin C$ we have $R'_a = R_a$. Thus, $x_a R_a (\mu'_a, p) R_a x'_a$ as $\mu'_a \in H_2 \cup H_3$ and $x'_a = (\mu'_a, p')$. Suppose that $\mu'_a \in H_3$. If $x_a = a$, then $a R_a (\mu'_a, p) P_a x'_a$, which contradicts that x' is individually rational. If $x_a \neq a$, then $(\mu_a, p') P_a (\mu_a, p) R_a x'_a$, which is a contradiction to $x' \in \mathcal{E}_{R'}$. Hence, $\mu'_a \notin H_3$, i.e., $\mu'_a \in H_2$. It now follows directly that if $x_a = a$ then $x_a I_a x'_a$, and if $x_a \neq a$ then $(\mu_a, p') P_a x'_a$, which is a contradiction to $x' \in \mathcal{E}_{R'}$. Hence, $x_a = a$, $\mu'_a \in H_2$, and $x_a I_a x'_a$. $\qquad\square$

Lemma 2. Let R and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{\mathcal{R}}$, and $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states where $x'_a P_a x_a$ for all $a \in C$. Let also $G = (a_1, \ldots, a_q)$ be a trading cycle from μ to μ', and $(\mu_{a_1}, \ldots, \mu_{a_q})$ the corresponding set of houses. Then $\mu_{a_k} \in H_1$ for some $a_k \in G$ implies that $\mu_{a_j} \notin H_3$ for all $a_j \in G$.

Proof. For notational simplicity, let $h_j = \mu_{a_j}$ for all $1 \leq j \leq q$. To obtain a contradiction, suppose that $h_k \in H_1$ but $h_l \in H_3$ where, without loss of generality, $k < l$ and $h_j \in H_2$ for all $k < j < l$. A first observation is that $a_j \notin C$ for $k \leq j < l$. This follows directly from Lemma 1(i) as $\mu'_{a_j} \in H_2 \cup H_3$ for all $k \leq j < l$ by construction. Hence, $R'_{a_j} = R_{a_j}$ for all $k \leq j < l$. We will consider the cases when $k + 1 = l$ and $k + 1 < l$. By Lemma 1(ii), $x_{a_k} = a_k = h_k$ and $h_{k+1} \in H_2$, implying that $k + 1 = l$ is impossible.

A second observation is that $x'_{a_j} I_{a_j} x_{a_j}$ for $k \leq j < l$. For $j = k$ this follows from Lemma 1(ii). For all $k < j < l - 1$, it follows from the facts that $h_j, h_{j+1} \in H_2$, $R'_{a_j} = R_{a_j}$, and that x and x' are equilibrium states. For $j = l - 1$, we have $a_{l-1} \notin C$, $\mu_{a_{l-1}} \in H_2$ and $\mu'_{a_{l-1}} \in H_3$. Applying Lemma 1(ii) to R' now yields $x'_{a_{l-1}} = a_{l-1}$ and $(h_{l-1}, p) I'_{a_{l-1}} a_{l-1}$. Now, because $k + 1 < l$ and both $x_{a_k} = a_k$ and $x'_{a_{l-1}} = a_{l-1}$, there exist l' and l'' where $k \leq l' < l'' \leq l - 1$ such that for all $l' < j < l''$ we have $x_j \neq a_j \neq x'_j$ and both $x_{a_{l'}} = a_{l'}$ and $x'_{a_{l''}} = a_{l''}$. But then:

(a) $x'_{a_{l'}} = (h_{l'+1}, p)$, $a_{l'} I_{a_{l'}} (h_{l'+1}, p)$ and $R'_{a_{l'}} = R_{a_{l'}}$,

20

(b) $(h_j, p) I_{a_j}(h_{j+1}, p)$ and $R'_{a_j} = R_{a_j}$ for all $l' < j < l''$,

(c) $(h_{l''}, p) I'_{a_{l''}} a_{l''}$ (where $R'_{a_{l''}} = R_{a_{l''}}$ if $l'' < l$).

Now by (a)–(c), houses $a_{l'}$ and $a_{l''}$ are connected by indifference at profile R' which contradicts that the profile R' belongs to $\tilde{\mathcal{R}}$. $\square$

An immediate consequence of Lemma 2 is that the set of trading cycles from μ to μ' can be partitioned into two disjoint groups as explained in the following definition.

Definition 5. Let R and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{\mathcal{R}}$, and $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states where $x'_a P_a x_a$ for all $a \in C$. Let $A^+ \subseteq A$ be such that $a \in A^+$ precisely when there is a trading cycle G from μ to μ' such that $a \in G$ and $\mu_{a'} \in H_1$ for some $a' \in G$. Let $A^- = A \setminus A^+$.

The notations A^+ and A^- are chosen because all agents in A^+ are weakly better off at the equilibrium state x' than at the equilibrium state x, while no agent in A^- is strictly better off at the equilibrium state x' than at the equilibrium state x. This is demonstrated in the next lemma.

Lemma 3. Let R and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{\mathcal{R}}$, and $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states where $x'_a P_a x_a$ for all $a \in C$. Let $G \subseteq A$ be a trading cycle from μ to μ'. If $x'_a P_a x_a$ for some agent $a \in G$, then $G \subseteq A^+$ and $x'_{\hat{a}} R_{\hat{a}} x_{\hat{a}}$ for all $\hat{a} \in G$.

Proof. Note first that Lemma 1(i) implies that $\mu'_a \in H_1$ since $x'_a P_a x_a$. Hence, $G \subseteq A^+$ and $\mu_{\hat{a}} \in H_1 \cup H_2$ for all $\hat{a} \in G$ by Lemma 2 and by the fact that G is a trading cycle from μ to μ'. Note next that either $x'_{\hat{a}} P_{\hat{a}} x_{\hat{a}}$ or $x_{\hat{a}} R_{\hat{a}} x'_{\hat{a}}$ for all $\hat{a} \in G$. In the latter case, $\hat{a} \notin C$ and $R'_{\hat{a}} = R_{\hat{a}}$. But then from $\mu_{\hat{a}} \in H_1 \cup H_2$ we obtain $x'_{\hat{a}} R_{\hat{a}}(\mu_{\hat{a}}, p') R_{\hat{a}}(\mu_{\hat{a}}, p)$ because x' is an equilibrium state. If $x_{\hat{a}} = (\mu_{\hat{a}}, p)$, then $x'_{\hat{a}} R_{\hat{a}} x_{\hat{a}}$. Otherwise, $x_{\hat{a}} = \hat{a}$ from the fact that x' is an equilibrium state, $x'_{\hat{a}} R_{\hat{a}} \hat{a}$. Hence, $x'_{\hat{a}} R_{\hat{a}} x_{\hat{a}}$ for all $\hat{a} \in G$. $\square$

Theorem 1. There is a unique minimum equilibrium price vector $p^* \in \Pi_R$ for each profile $R \in \tilde{\mathcal{R}}$.

Proof. Let $R \in \tilde{\mathcal{R}}$, and let $x' = (\mu', \nu', p') \in \mathcal{E}_R$ and $x'' = (\mu'', \nu'', p'') \in \mathcal{E}_R$ be two equilibrium states. We will demonstrate that $x = (\mu, \nu, p) \in \mathcal{E}_R$ for some assignments μ and ν if $p_h = \min\{p'_h, p''_h\}$ for each $h \in H$. The proof of the theorem then follows directly since the set Π_R is closed and bounded from below.

Consider the equilibrium states $x' = (\mu', \nu', p')$ and $x'' = (\mu'', \nu'', p'')$, and the trading cycles from μ' to μ''. Let the sets A^+ and A^- be defined as in Definition 5 with the restriction $R = R'$, and define $p_h = \min\{p'_h, p''_h\}$ for each $h \in H$. Let:

$$\mu_a = \begin{cases} \mu''_a \text{ if } a \in A^+, \\ \mu'_a \text{ if } a \in A^-. \end{cases}$$

21

Note that the assignment μ defined in this way becomes bijective because no agent $a \in A^+$ belongs to a trading cycle containing an agent in A^- by Lemma 2. Let now $x_a = x_a''$ and $\nu_a = \nu_a''$ for all $a \in A^+$, and $x_a = x_a'$ and $\nu_a = \nu_a'$ for all $a \in A^-$. To prove the theorem, we need to demonstrate that $x = (\mu, \nu, p)$ is an equilibrium state. Now, the following is true for any $a \in A^+$ (recall that $R = R'$):

(a) $x_a'' R_a a$ because of individual rationality,

(b) $x_a'' R_a (h, p'')$ for all $h \in H$ as x'' is an equilibrium state,

(c) $x_a'' R_a x_a'$ by Lemma 3 since $a \in A^+$,

(d) $x_a' R_a (h, p')$ for all $h \in H$ as x' is an equilibrium state.

From (a)–(d) in the above and the construction that $x_a = x_a''$ and $\nu_a = \nu_a''$ for all $a \in A^+$, it now follows that $x_a R_a a$ and $x_a R_a (h, p)$ for all $a \in A^+$ and all $h \in H$. Symmetric arguments now give that $x_a R_a a$ and $x_a R_a (h, p)$ for all $a \in A^-$ and all $h \in H$. Hence, $x = (\mu, \nu, p)$ is an equilibrium state if the assignment ν implies maximal trade (see Definition 1). If this condition is not satisfied, it only remains to change the assignment ν so trade becomes maximal. Hence, $x = (\mu, \nu, p)$ is an equilibrium state. $\qquad\square$

Proposition 5. Let f be a minimum price mechanism with domain $\tilde{\mathcal{R}}$. Then f is a maximum trade rule.

Proof. To obtain a contradiction, suppose that f is a minimum price mechanism but not a maximum trade rule. Let also the state $x' = (\mu', \nu', p') \in \mathcal{E}_R$ be selected by the minimum price mechanism, i.e., p' is the unique minimum equilibrium price vector in Π_R by Theorem 1. From Theorem 1 and the assumption that f is not a maximum trade rule, it then follows that there is a state $x = (\mu, \nu, p) \in \mathcal{E}_R$ where $p' \neq p$, $p_h' \leq p_h$ for all $h \in H$, and $|\nu'| < |\nu|$.

Recall next that the complete trade from μ to μ' can be decomposed uniquely into a number of trading cycles. Because $|\nu'| < |\nu|$, by assumption, this means that there must be a trading cycle $G = (a_1, \ldots, a_q)$ from μ to μ' where $\Sigma_{a \in G} \nu_a > \Sigma_{a \in G} \nu_a'$, and, consequently, an agent $a_l \in G$ with $\mu_{a_l}' = a_l$ and $\nu_{a_l}' = 0$. Note also that $\mu_{a_j} \in H_1 \cup H_2$ for all $j \in G$ as $p_h' \leq p_h$ for all $h \in H$.

Let now agent $a_l \in G$ be chosen as above, and let $a_j \in G$ for all $1 \leq j \leq q$. We will next demonstrate that $\mu_{a_l} \in H_2$ and $a_l I_{a_l}(\mu_{a_l}, p)$. As $\mu_{a_l} \in H_1 \cup H_2$, it suffices to show that $\mu_{a_l} \notin H_1$ to prove the first part of the statement. To obtain a contradiction, suppose that $\mu_{a_l} \in H_1$. Because $\mu_{a_l} \in H_1$, $\mu_{a_l} \neq a_l$, and $(\mu, \nu, p) \in \mathcal{E}_R$, it follows that:

$$(\mu_{a_l}, p') P_{a_l}(\mu_{a_l}, p) R_{a_l} a_l.$$

But then state (μ', ν', p') cannot belong to $\mathcal{E}_R$ since $x_{a_l}' = a_l$. Hence, $\mu_{a_l} \in H_2$. But if $\mu_{a_l} \in H_2$ and $x_{a_l}' = a_l$, it is immediately clear that $x_{a_l}' I_{a_l} x_{a_l}$ as both x and x' belong to $\mathcal{E}_R$. Note that the latter condition may also be written as $a_l I_{a_l}(\mu_{a_l}, p)$ since $x_{a_l}' = a_l$.

Let again agent $a_l \in G$ be defined as in the above. Given the above findings, we next remark that either (i) $\mu_{a_j} \in H_2$ for all $k < j \leq l$ and $\mu_{a_k} \in H_1$ for some $a_k \in G$, or (ii)

22

$\mu_{a_j} \in H_2$ for all $a_j \in G$. We will demonstrate that both these cases lead to the desired contradiction.

Case (i). Suppose first that $x_{a_k} \neq a_k$. Because $(\mu, \nu, p) \in \mathcal{E}_R$, $\mu_{a_k} \in H_1$, and $\mu'_{a_k} = \mu_{a_{k+1}} \in H_2$ it follows that:
$$(\mu_{a_k}, p') P_{a_k}(\mu_{a_k}, p) R_{a_k} x'_{a_k},$$
which contradicts that $(\mu', \nu', p') \in \mathcal{E}_R$. Hence, $x_{a_k} \neq a_k$ cannot be the case. Suppose instead that $x_{a_k} = a_k$, and note that $x_{a_j} I_{a_j} x'_{a_j}$ for all $k \leq j \leq l - 1$. This follows as both x and x' belong to $\mathcal{E}_R$ and $\mu_{a_j} \in H_2$ for all $k < j \leq l$. But then houses a_k and a_l are connected by indifference at prices p as $a_k I_{a_k}(a_{k+1}, p)$ and $a_l I_{a_l}(\mu_{a_l}, p)$. Hence, $x_{a_k} = a_k$ cannot be the case.

Case (ii). If $\mu_{a_j} \in H_2$ for all $a_j \in G$, it follows, by the same arguments as in Case (i), that $x_{a_j} I_{a_j} x'_{a_j}$ for all $a_j \in G$. Now, if $\mu_{a_{l'}} = a_{l'}$ for some $a_{l'} \in G$ and $\mu_{a_j} \neq a_j$ for all $l' < j \leq l$, houses $a_{l'}$ and a_l are connected by indifference at prices p as $a_{l'} I_{a_{l'}}(a_{l'+1}, p)$ and $a_l I_{a_l}(\mu_{a_l}, p)$. On the other hand, if $\mu_{a_{l'}} \neq a_{l'}$ for all $a_{l'} \in G$, then the trade at state x' can be increased since $\mu'_{a_l} = a_l$ and $\nu'_{a_l} = 0$, which contradicts that $x' \in \mathcal{E}_R$. $\square$

For any assignment μ, let $|\mu| = |\{a \in A : \mu_a \neq a\}|$ denote the number of agents who do not stay in their house (or who do not exercise their outside option). For convenience, we will call $|\mu|$ the cardinality of μ. The following lemma shows that for profiles in $\tilde{\mathcal{R}}$, it suffices to maximize the cardinality of $|\mu|$ instead of the cardinality of $|\nu|$.

Lemma 4. Let $R \in \tilde{\mathcal{R}}$ and $x = (\mu, \nu, p)$ be a state such that for all $a \in A$: (i) $x_a R_a a$ and (ii) $x_a R_a(h, p)$ for all $h \in H$. If the cardinality of $|\mu|$ is maximal among all states satisfying (i) and (ii) with price vector p, then x is an equilibrium state, i.e., $|\nu|$ is maximal among all states satisfying (i) and (ii) with price vector p.

Proof. To obtain a contradiction, suppose that $|\nu|$ is maximal among all states satisfying (i) and (ii) with price vector p. Then there exists an equilibrium state $x' = (\mu', \nu', p')$ with $p' = p$ and $|\nu'| > |\nu|$. Then for some $a \in A$, $\nu'_a = 1$ and $\nu_a = 0$. Thus, $\mu_a = a$ and $x_a = a$. If $\mu'_a = a$, then $(a, \underline{p}_a) I_a a$ which contradicts our assumption $\nu_a = 0$ only if $a P_a(a, \underline{p}_a)$. Thus, $\mu'_a \neq a$ and a belongs to some trading cycle $G = (a_1, \ldots, a_q)$ from μ to μ' where $a_1 = a$ and $\mu'_{a_j} = \mu_{a_{j+1}}$. Note that $a I_a(\mu'_a, p)$. If for all $1 \leq j \leq q$, $\mu'_{a_j} \neq a_j$, then (μ, ν, p) could not have maximized $|\mu|$ (simply define $x'' = (\mu'', \nu'', p'')$ by $p'' = p$, $x''_{a''} = x_{a''}$ for all $a'' \in A \backslash G$, and $x''_{a''} = x'_{a''}$ for all $a'' \in G$; then $|\mu''| > |\mu|$), which is a contradiction. Thus, choose $1 < l \leq q$ minimal such that $\mu'_{a_l} = a_l$. Note that $\mu_{a_l} \neq a_l$. But then $a_1 I_{a_1}(\mu'_{a_1}, p)$, $(\mu_{a_j}, p) I_{a_j}(\mu_{a_{j+1}}, p)$ for all $1 < j < l$, and $(\mu_{a_l}, p) I_{a_l} a_l$, which contradicts the fact that R is not connected by indifference. $\square$

Note that for any two equilibrium states $x = (\mu, \nu, p)$ and $x' = (\mu', \nu', p')$ with $p = p'$, it holds that:
$$x_a I_a x'_a \text{ for all } a \in A. \tag{1}$$

Condition (1) together with Lemma 4 have the important consequence that without loss of generality, *for profiles $R \in \tilde{\mathcal{R}}$ below in Lemma 5 we focus on equilibrium states $x = (\mu, \nu, p)$*

where the cardinality of $|\mu|$ is maximized. Lemma 5 is the key in the proofs of Theorems 2 and 3.

Lemma 5. Let $R, R' \in \widetilde{\mathcal{R}}$ be two profiles such that $R' = (R'_C, R_{-C})$ for some $C \subset A$. Let also $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states such that $x'_a P_a x_a$ for all $a \in C$ and $|\mu|$ is maximized among all equilibrium states with price vector p. If $H_1 \neq \emptyset$, then there is a subset $S \subseteq H_1$ such that $A_S = \emptyset$ where:

$$A_S = \{a \in A : \mu_a \notin S \text{ and } x_a I_a(h, p) \text{ for some } h \in S \backslash \{a\}\}.$$

Proof. Without loss of generality, we may assume that $C = \{a \in A : x'_a P_a x_a\}$ as $R'_a = R_a$ is an allowed report for all agents $a \in C$. To obtain a contradiction, suppose that $A_S \neq \emptyset$ for each $S \subseteq H_1$. Then $A_{H_1} \neq \emptyset$ and there is an agent $a_0 \in A$ with $\mu_{a_0} \notin H_1$, $x_{a_0} I_{a_0}(h, p)$ for some $h \in H_1$, and $h \neq a_0$. Now, $x_{a_0} R_{a_0} a_0$ by individual rationality and $(h, p') P_{a_0}(h, p)$ as preferences are strictly monotonic, $h \neq a_0$ and $p'_h < p_h$. Hence, $(h, p') P_{a_0}(h, p) I_{a_0} x_{a_0} R_{a_0} a_0$ and, obviously, $a_0 \in C$ and $\mu'_{a_0} \in H_1$.

Note next that agent a_0 belongs to some trading cycle $G = (a_t, a_{t-1}, \ldots, a_1, a_0)$ from μ to μ'. Since $\mu'_{a_0} \in H_1$, it follows from Lemma 2 that $\mu_{a_j} \in H_1 \cup H_2$ for all $0 \leq j \leq t$, and $\mu_{a_0} \in H_2$ because $\mu_{a_0} \notin H_1$. Let $0 < l \leq t$ be minimal such that $\mu_{a_l} \in H_1$, and note that such an index exists because $\mu'_{a_0} = \mu_{a_t} \in H_1$. But then $\mu'_{a_l} \in H_2$ and, consequently, $a_l \notin C$ by Lemma 1(i). Because $\mu_{a_l} \in H_1$ and $\mu'_{a_l} \in H_2$, Lemma 1(ii) yields $x_{a_l} = a_l$ and $a_l I_{a_l}(\mu_{a_{l-1}}, p)$. Furthermore, for $0 < j < l$, we have $\mu'_{a_j} \in H_2$, $a_j \notin C$, $x_{a_j} I_{a_j} x'_{a_j}$ and $x'_{a_j} \neq a_j$ because R belongs to $\widetilde{\mathcal{R}}$.

It is next established that $x_{a_j} P_{a_j}(a_l, p)$ for all $0 \leq j < l$. To obtain a contradiction, suppose that $x_{a_0} I_{a_0}(a_l, p)$. Then it is possible to define an assignment μ'' such that $\mu''_a = \mu_a$ for $a \in A \backslash \{a_0, a_1, \ldots, a_l\}$, $\mu''_{a_j} = \mu_{a_{j-1}}$ for $0 < j \leq l$, and $\mu''_{a_0} = a_l$. Further, let $\nu''_a = \nu_a$ for $a \in A \backslash \{a_0, a_1, \ldots, a_l\}$ and $\nu''_a = 1$ for $a \in \{a_0, a_1, \ldots, a_l\}$. Note that $\nu''_a = 1$ for $a \in \{a_0, a_1, \ldots, a_l\}$ is a utility maximizing choice for agent a since for $a = a_j$, $x_a R_a a$, $x_a R_a(h, p)$ for all $h \in H$, $x_{a_j} I_{a_j}(\mu_{a_{j-1}}, p)$, and both $x'_{a_j} \neq a_j$ and $\nu'_{a_j} = 1$. Then the triple (μ'', ν'', p) satisfies the requirements of Definition 1. However, by comparing $x'' = (\mu'', \nu'', p)$ and $x = (\mu, \nu, p)$, we see that $|\mu''| > |\mu|$, i.e, $\mu_{a_l} = a_l$ while $\mu''_{a_l} \neq a_l$ (and $\mu''_{a_j} = \mu'_{a_j} \neq a_j$ for $0 \leq j < l$). This contradicts that x is an equilibrium state where the cardinality of μ is maximized. Similar argument can be used to derive a contradiction if $x_{a_j} I_{a_j}(a_l, p)$ for some $0 < j < l$. Hence, $x_{a_j} P_{a_j}(a_l, p)$ for all $0 \leq j < l$.

Recall next that $A_S \neq \emptyset$ for each $S \subseteq H_1$ by assumption. Then because $a_l \in H_1$ and $x_{a_l} = a_l$, it follows that $A_{\{a_l\}} \neq \emptyset$. Let $\hat{a}_0 \in A_{\{a_l\}}$, and note that $\hat{a}_0 \notin \{a_0, a_1, \ldots, a_l\}$ as $x_{a_j} P_{a_j}(a_l, p)$ for all $0 \leq j < l$ by the above conclusion. Consider next the following two cases:

(I) Suppose that $\mu_{\hat{a}_0} \notin H_1$. Then, again, $\hat{a}_0$ belongs to some trading cycle $\hat{G} = (\hat{a}_t, \hat{a}_{t-1}, \ldots, \hat{a}_1, \hat{a}_0)$ from μ to μ'. Again $\hat{a}_0 \in C$ and $\mu_{\hat{a}_0} \in H_2$, and there exists $0 < k \leq t$ such that $\mu_{\hat{a}_k} \in H_1$ and both $\mu_{\hat{a}_j} \in H_2$ and $x'_{\hat{a}_j} \neq \hat{a}_j$ for all $0 < j < k$. Again $x_{\hat{a}_k} = \hat{a}_k$ and $\hat{a}_k I_{\hat{a}_k}(\mu'_{\hat{a}_k}, p)$. Because G and $\hat{G}$ are trading cycles, we have

$\hat{a}_j \notin \{a_0, a_1, \dots, a_l\}$ for all $0 \leq j \leq k$. Now if (a) $\hat{a}_k = \mu'_{a_0}$ or (b) for some $0 \leq j < k$, $x_{\hat{a}_j} I_j(\hat{a}_k, p)$ or (c) for some $0 \leq j \leq l$ and $0 \leq j' \leq k$, we have $x_{a_j} I_{a_j}(\mu_{\hat{a}_{j'}}, p)$, then, in all cases (a), (b) and (c), we can construct similarly as above from x an equilibrium state x'' where the cardinality of μ'' is larger compared to the cardinality of μ, which is a contradiction. Thus, suppose that (a), (b) and (c) are not true, in particular, $x_{a_j} P_{a_j}(\hat{a}_k, p)$ for all $0 \leq j \leq l$, and $x_{\hat{a}_j} P_{\hat{a}_j}(\hat{a}_k, p)$ for all $0 \leq j < k$. By $\hat{a}_k \in H_1$ and the above assumption that $A_S \neq \emptyset$ for each $S \subset H_1$, it then follows that $A_{\{\hat{a}_k\}} \neq \emptyset$. Let $a'_0 \in A_{\{\hat{a}_k\}}$. Now we have $a'_0 \notin \{\hat{a}_0, \hat{a}_1, \dots, \hat{a}_k\} \cup \{a_0, a_1, \dots, a_l\}$.

(II) Suppose that $\mu_{\hat{a}_0} \in H_1$. Now, if $x_{a_j} I_{a_j}(\mu_{\hat{a}_0}, p)$ for some $0 \leq j \leq l$, it is possible to construct, in a similar fashion as in the above, an equilibrium state x'' where the cardinality of μ'' trade is larger compared to the cardinality of μ, which is a contradiction. Thus, $x_{a_j} P_{a_j}(\mu_{\hat{a}_0}, p)$ for all $0 \leq j \leq l$. By $\mu_{\hat{a}_0} \in H_1$ and the above assumption that $A_S \neq \emptyset$ for each $S \subseteq H_1$, it then follows that $A_{\{\mu_{\hat{a}_0}\}} \neq \emptyset$. Let $a'_0 \in A_{\{\mu_{\hat{a}_0}\}}$. Since $x_{a_j} P_{a_j}(\mu_{\hat{a}_0}, p)$ for all $0 \leq j \leq l$, we have $a'_0 \notin \{\hat{a}_0, a_0, a_1, \dots, a_l\}$.

We next observe that if $\mu_{a'_0} \notin H_1$, then as in (I) we can find another sequence $a'_0, a'_1, \dots, a'_o$ where $x_{a'_o} = a_o \in H_1$, $\mu_{a'_j} \in H_2$, $\mu_{a'_j} = \mu_{a'_{j-1}}$, $x'_{a'_j} \neq a'_j$ for $0 < j < o$, and $\mu'_{a'_o} = \mu_{a'_{o-1}}$; and otherwise as in (II) $\mu_{a'_0} \in H_1$. Then either we can use similar arguments as above to construct another equilibrium state x'' from x where the cardinality of μ'' trade is larger compared to the cardinality of μ or otherwise we continue to construct another (disjoint) sequence as in (I) or (II) (where in (II) the sequence consists of one agent) which eventually leads to a contradiction because in any such sequence there is a new agent $a \in A$, who does not belong to any of the previously identified sequences, with $\mu_a \in H_1$. The contradiction then follows as H_1 is finite. $\qquad \square$

Theorem 2. Let f be a minimum price mechanism with domain $\tilde{\mathcal{R}}$. Then f is group non-manipulable.

Proof. To obtain a contradiction, suppose that some nonempty group $C \subseteq A$ can manipulate the minimum price mechanism f at a profile $R \in \tilde{\mathcal{R}}$ by reporting preferences $R' = (R'_C, R_{-C}) \in \tilde{\mathcal{R}}$. More precisely, let $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states such that $x'_a P_a x_a$ for all $a \in C$. By condition (1) and as in the proof of Lemma 5, we will, without loss of generality, assume that $C = \{a \in A : x'_a P_a x_a\}$ and the cardinality of μ is maximized among all equilibrium states with price vector p. Then $\mu'_a \in H_1$ for all $a \in C$ by Lemma 1(i) as $x'_a P_a x_a$ for all $a \in C$ and $C \neq \emptyset$. Hence, $H_1 \neq \emptyset$ and $p_h > p'_h \geq \underline{p}_h$ for all $h \in H_1$. Let now:

$$A_S = \{a \in A : \mu_a \notin S \text{ and } x_a I_a(h, p) \text{ for some } h \in S \setminus \{a\}\}.$$

From Lemma 5, it follows that there exists a nonempty set $S \subseteq H_1$ such that $A_S = \emptyset$. But then it is possible to decrease p_h for all $h \in S$ (since $p_h > p'_h \geq \underline{p}_h$ for all $h \in S$, and if $h = a \in S$ and $\mu_a \notin S$, then agent a is not affected by the decrease because $x_a R_a a$ and $p_a > \underline{p}_a$) and obtain a new equilibrium state by the Perturbation Lemma in Alkan et

al. (1991) which contradicts that state x is selected by f at profile R. Hence, f is group non-manipulable. $\qquad\square$

Proposition 3. A minimum price mechanism f is non-manipulable on the domain $\mathcal{R}$ if and only if $1 \leq |A| \leq 3$.

Proof. We first prove that a minimum price mechanism is non-manipulable *if* $1 \leq |A| \leq 3$. We will, however, only prove the result for $|A| = 3$ as the proof for $1 \leq |A| \leq 2$ is a special case of the proof for $|A| = 3$. To obtain a contradiction, suppose that $|A| = 3$ and that some nonempty group $C \subseteq A$ can manipulate a minimum price mechanism f at a profile $R \in \tilde{\mathcal{R}}$ by reporting preferences $R' = (R'_C, R_{-C}) \in \tilde{\mathcal{R}}$. Let also $x = (\mu, \nu, p)$ and $x' = (\mu', \nu', p')$ represent the selections of f at profiles R and R', respectively. As in the proof of Lemma 5, we will, without loss of generality, assume that $C = \{a \in A : x'_a P_a x_a\}$. Note also that $\mu'_a \in H_1$ for all $a \in C$ by Lemma 1(i) as $x'_a P_a x_a$ for all $a \in C$.

We need to demonstrate that $|H_1| = 3$. The conclusion then follows by the Perturbation Lemma in Alkan et al. (1991) in the same fashion as in the proof of Theorem 2. Note first that $|H_1| \geq 1$, by the above observation, as $C \neq \emptyset$. To obtain a contradiction to $|H_1| = 3$, suppose that $|H_1| = 2$. This also means that there is an agent $a \in A$ with $\mu'_a \notin H_1$. Suppose next, without loss of generality, that $a = 3$, and note that $3 \notin C$. If $\mu_3 = \mu'_3$, then it must be the case that $x_3 P_3(\mu_j, p)$ for $j = 1, 2$, otherwise x' cannot be an equilibrium since $\mu_j \in H_1$ for $j = 1, 2$. But in this case, is possible to decrease the prices of houses $\mu_1 \in H_1$ and $\mu_2 \in H_1$ at state x, by the Perturbation Lemma in Alkan et al. (1991), and obtain a new equilibrium which contradicts that p is a minimum equilibrium price vector at profile R. Hence, $\mu_3 \neq \mu'_3$, and consequently, $\mu_3 \in H_1$ as $\mu'_3 \notin H_1$ and $|H_2| = 2$. But this also means that $\mu_3 = 3$, $\mu'_3 \in H_2$, and $x_3 I_3(\mu'_3, p)$, otherwise, state x' cannot be an equilibrium as $\mu_3 \in H_1$.

Suppose next, without loss of generality, that $\mu'_3 = 2$, i.e., that $1, 3 \in H_1$. Hence, $x_3 I_3(2, p)$, by the above conclusion, and $x_3 P_3(1, p)$ otherwise x' cannot be an equilibrium at profile R as $1 \in H_1$. But then it must be the case that $x_j I_j(1, p_1)$ for the agent $j \neq 3$ with $\mu_j \neq 1$ otherwise it is possible to decrease $p_1 \in H_1$ and obtain a new equilibrium which contradicts that p is a minimum equilibrium price vector at profile R. By identical arguments, it must also be the case that $x_k I_k(3, p_3)$ some agent $k \neq 3$. If $\mu_k = 2$, agents k and 3 can swap houses at allocation x and a new equilibrium is obtained, but it is also possible to decrease the prices of all houses but 2 and obtain a new equilibrium, by the Perturbation Lemma in Alkan et al. (1991), which contradicts that p is a minimum equilibrium price vector at profile R. If $\mu_k = 1$, a new equilibrium can be obtained at prices p and for $\mu''_j = 1$, $\mu''_k = 3$, and $\mu''_3 = 2$, and it is again possible to decrease the prices of all houses but 2 to obtain a new equilibrium at profile R. Hence, in both cases it is possible to obtain a contradiction to $|A| = 2$. Hence, $|H_1| = 3$ as desired.

We next prove that a minimum price mechanism f is non-manipulable on the domain $\mathcal{R}$ *only if* $1 \leq |A| \leq 3$, i.e., that f can be manipulated by some agent, at some profile $R \in \mathcal{R}$, whenever $|A| > 3$. The proof is based on Example 2, and by using the same arguments as in the proof of Proposition 2, it is sufficient to demonstrate the result for $|A| = 3$.

Consider now the state $\hat{x} = (\hat{\mu}, \hat{\nu}, \hat{p}) \in \mathcal{E}_R$ and suppose that $\hat{p}$ is a minimum equilibrium price vector. We first demonstrate that $\hat{p} = p' = (1, 0, 0, 0)$ or $\hat{p} = p'' = (0, 1, 0, 0)$. From Example 2, it is clear that either $\hat{p}_1 < 1$ or $\hat{p}_2 < 1$ as $\hat{p}$ is a minimum equilibrium price vector. Suppose that $\hat{p}_1 < 1$. Then $\hat{\mu}_3 = 1$ by envy-freeness as $\hat{x}$ is an equilibrium. Consequently, $\hat{\mu}_1 = 3$, $\hat{p}_3 = 0$ by individual rationality for agent 1, and it then follows that $\hat{\mu}_2 = 2$ by individual rationality for agent 2. But then it must be the case that $\hat{\mu}_4 = 4$ and $\hat{p}_2 \geq 1$ because otherwise agent 4 will envy agent 2 at state $\hat{x}$. Hence, $\hat{p} \geq p''$. But then $\hat{p} = p''$, by definition, as $\hat{p}$ is a minimum equilibrium price vector by assumption. Analogous arguments can be used to show that $\hat{p} = p'$ if $\hat{p}_2 < 1$.

Let now f be minimum price mechanism on domain $\mathcal{R}$. Then f chooses either p' or p''. If f chooses p', then agent 3's utility is equal to $v_{34} - p_4' = 1$. Let R' denote the profile of quasi-linear preferences where the entry v_{32} in the matrix V from Example 2 is replaced by $v_{32}' = 2$. Obviously, $x' \notin \mathcal{E}_{R'}$ because $(2, p_2')P_3'x_3'$. On the other hand, it is easy to check that $x'' \in \mathcal{E}_{R'}$. Also, p'' is the unique minimum equilibrium price vector at profile R'. To see this, suppose that $\hat{x} = (\hat{\mu}, \hat{\nu}, \hat{p}) \in \mathcal{E}_{R'}$ and that $\hat{p} \neq p''$ is a minimum equilibrium price vector at profile R'. Then $\hat{p}_2 < 1$, which implies that $\hat{\mu}_4 = 2$ and $\hat{\mu}_3 = 1$. But then individual rationality cannot be satisfied for both agents 1 and 2 at state x''. Thus, p'' must be chosen by f for profile R'. Then, by individual rationality for agents 1 and 2, it follows that agent 3 must receive house 1. Because $R' = (R_3', R_{-3})$ and agent 3's utility from $(1, p'')$ under R_3 is equal to $v_{31} - p_1'' = 2 > 1$, agent 3 can profitably manipulate f at R.

If f chooses p'', it can be shown, by identical arguments as in the above, that agent 4 can manipulate the rule by replacing the entry v_{41} in the matrix V from Example 2 is by $v_{41}' = 2$. $\qquad\square$

Proposition 6. For any given profile $R \in \mathcal{R}$, a Dutch price sequence $(p^t)_{t=1}^T$ contains a finite number of price vectors, i.e., $T < \infty$.

Proof. To obtain a contradiction, suppose that $(p^t)_{t=1}^T$ is a Dutch price sequence with $T = \infty$. Let also $(x^t)_{t=1}^T$ be a supporting sequence of equilibrium states, where $x^t = (\mu^t, \nu^t, p^t)$. Because there is only a finite number of assignments, there is an infinite subsequence $(t_j)_{j=1}^{\infty}$ of steps such that for some finite number j' and some assignments μ' and ν', it holds that $\mu^{t_j} = \mu'$ and $\nu^{t_j} = \nu'$ for all $j \geq j'$. Let also $p^t \to p^e$ as $t \to \infty$. We will demonstrate that $p^e \in \Pi^{\nu'}$, which contradicts that $T = \infty$ since $\nu^{t_{j'}} = \nu'$.

Consider now the states $x^{t_j} = (\mu^{t_j}, \nu^{t_j}, p^{t_j}) = (\mu', \nu', p^{t_j})$ for $j \geq j'$. Because $x^{t_j} \in \mathcal{E}$, the following holds for all $a \in A$:

$$x_a^{t_j} R_a a \text{ and } x_a^{t_j} R_a(h, p^{t_j}) \text{ for all } j \geq j' \text{ and for all } h \in H.$$

Moreover, by continuity, the following holds for all $a \in A$:

$$(\mu', \nu', p^e) R_a a \text{ and } (\mu', \nu', p^e) R_a(h, p^e) \text{ for all } h \in H.$$

But then $p^e \in \Pi^{\nu'}$. $\qquad\square$

Theorem 3. For any given profile $R \in \tilde{\mathcal{R}}$, a Dutch price sequence $(p^t)_{t=1}^T$ is unique and p^T is the unique minimum equilibrium price vector in Π_R.

Proof. We first prove that a Dutch price sequence is unique. To do this, consider the profile $R \in \tilde{\mathcal{R}}$, and let $x' = (\mu', \nu', p) \in \mathcal{E}_R$ and $x'' = (\mu'', \nu'', p) \in \mathcal{E}_R$ be two equilibrium states with a common price vector p (recall from Theorem 1 that the minimum price vector is unique on the domain $\tilde{\mathcal{R}}$). We need to demonstrate that $\nu' = \nu''$ to complete the proof. To see this, consider the correspondence ξ from the definition of a Dutch price sequence. If $\nu' = \nu''$, then there is just *one* state in $\xi(p)$. This also means that, for a given profile $R \in \tilde{\mathcal{R}}$, there is just one Dutch price sequence.

Suppose now that $\nu' \neq \nu''$, and consider a trading cycle $G = (a_1, \ldots, a_q)$ from μ' to μ'' such that $\nu'_{a_1} = 0$. Such a trading cycle exists since $\nu' \neq \nu''$ by assumption. For each agent a_j in the trading cycle G, let $r_j = (v'_{a_j}, v''_{a_j})$. Then r_j is equal to $(0,1)$, $(1,1)$ or $(0,1)$, but not $(0,0)$ as agent a_j is included in G. Moreover, $r_1 = (0,1)$ by assumption. Consider then the sequences $(r_l, r_{l+1}, \ldots, r_{q'})$ for $1 \leq l \leq q' \leq q$. Assume first that $r_j \neq (1,0)$ for all j, i.e., that $r_j = (0,1)$ or $r_j = (1,1)$ for all j. This means that trade is not maximal at the state x'. Note also that $x'_{a_j} I_{a_j} x''_{a_j}$ for all j since prices are the same at the two states x' and x''. Hence, there must be a q' such that $r_{q'} = (1,0)$. But since $r_1 = (0,1)$, there must also be an l such that $r_l = (0,1)$ and $r_j = (1,1)$ for $l < j < q'$. In that case, houses μ'_{a_l} and $\mu''_{a_{q'}}$ are connected by indifference, contradicting our assumption that $R \in \tilde{\mathcal{R}}$. Hence, $\nu' = \nu''$ must be the case, and, consequently, a Dutch price sequence is unique.

We next prove that p^T is the unique minimum equilibrium price vector p^* in Π_R. From the first part of this theorem and Proposition 6 we know that a Dutch price sequence is unique and finite. Suppose that it converges to the price vector p^e, but that $p^* \leq p^e$ and $p^* \neq p^e$. Let $x^e = (\mu^e, \nu^e, p^e)$ and $x^* = (\mu^*, \nu^*, p^*)$ be two corresponding equilibrium states in $\mathcal{E}_R$. Then $H_1 \neq \emptyset$, so $A(S) = \emptyset$ for some $S \subset H_1$ by Lemma 5. Note also that $S \neq \emptyset$ and $p^e \in \Pi^{\nu^e}$. Consider now the set of houses S and the set of agents $\hat{A}$ that are assigned houses in S, i.e., $\hat{A} = \{a \in A : \mu_a \in S\}$. Let also $A^0 = \{A \in \hat{A} : \nu^e = 0\}$ and $A^1 = \{A \in \hat{A} : \nu^e = 1\}$.

Since $p^* \leq p^e$ and $p^* \neq p^e$, for each $\varepsilon > 0$, there is a state $x^\varepsilon = (\mu^\varepsilon, \nu^\varepsilon, p^\varepsilon) \in \mathcal{E}_R$ such that $p^m \leq p^\varepsilon \leq p^e$, $p^m \neq p^\varepsilon \neq p^e$. Moreover, since $A(S) = \emptyset$, $x^\varepsilon_a = x^e_a$ can be chosen for $a \in A \setminus \hat{A}$ for "sufficiently small" ε. Since there is only a finite number of assignments ν, there is an infinite and increasing sequence $(\varepsilon_j)_{j=1}^\infty$ such that $\nu^{\varepsilon_j} = \nu^*$ is constant for all j, and $\varepsilon_j \to 0$ as $j \to \infty$. Hence, $x^{\varepsilon_j} = (\mu^{\varepsilon_j}, \nu^*, p^{\varepsilon_j})$. But then $x^* = (\mu^*, \nu^*, p^e) \in \xi(x^{T-1})$ for some μ^* by continuity of preferences. But then there is a $p \in \Pi^{\nu^*}$ such that $p \leq p^e$, $p \neq p^e$. This means that the Dutch price sequence cannot stop at T, which contradicts our assumptions. Hence, $p^e = p^*$ must be the case. $\qquad\square$

Proposition 7 For any profile $R \in \mathcal{R}$, the minimum price mechanism generates a weakly higher revenue to the public authority compared to the current U.K. system.

Proof. The current U.K. system can, for any profile $R \in \mathcal{R}$, be represented by a state $x = (\mu, \nu, p)$ where $p_h = \infty$ for all $h \in H$ as the interpretation of this situation, due to the assumption that each house in H is bounded desirable, is that each tenant $a \in A$ is given

28

a take-it-or-leave-it offer either to buy house $h = a$ at price $p_h = \underline{p}_h$, or to continue renting house $h = a$.

Suppose now that the state $x' = (\mu', \nu', p')$ is selected by the minimum price mechanism at profile $R \in \mathcal{R}$. Note first that if $\nu_a = 1$, then $\nu'_a = 1$. This follows trivially because if $\nu_a = 1$ but $\nu'_a = 0$, then $(a, \underline{p}_a) R_a a$ and $a P_a (a, \underline{p}_a)$, respectively, which is a logical contradiction. This also means that all agents $a \in A$ that buy a house at state x also buy a house at state x'. Furthermore, all agents that buy a house at state x belong to one of the following two sets:

$$
\begin{aligned}
S &= \{a \in A : \mu_a = \mu'_a \text{ and } \nu_a = 1\}, \\
T &= \{a \in A : \mu_a \neq \mu'_a \text{ and } \nu_a = 1\}.
\end{aligned}
$$

It is clear that the revenue for the public authority from the sales to the agents in S is identical at states x' and x since $\mu_a = a = \mu'_a$ in this case. Because each agent in T must be involved in a trading cycle from μ to μ' and $p' \geq p$, it is clear that the revenue for the public authority from the sales to the agents in T is weakly higher at state x' than at state x. This proves the statement. Note that there may be agents $a \in T$ with $\nu_a = 0$ but $\nu'_a = 1$ which further increases the revenue for the public authority by switching from state x to state x'. $\qquad\square$

References

Alkan, A., Demange, G., Gale, D., 1991. Fair allocation of indivisible objects and criteria of justice. Econometrica 59, 1023–1039.

Andersson, T., Svensson, L.-G., 2014. Non-manipulable house allocation with rent control. Econometrica 82, 507–539.

Bogomolnaia, A., Moulin, H., 2004. Random matching under dichotomous preferences. Econometrica 72, 257–279.

Demange, G., Gale, D., 1985. The strategy structure of two-sided matching markets. Econometrica 53, 873–888.

Demange, G., Gale, D., Sotomayor, M., 1986. Multi-item auctions. J Polit Economy 94, 863–872.

Foley, D., 1967. Resource allocation and the public sector. Yale Econ Essays 7, 45–98.

Gale, D., 1984. Equilibrium in a discrete exchange economy with money. Int J Game Theory 13, 61–64.

Gale, D., Shapley, L.S., 1962. College admissions and the stability of marriage. Am Math Monthly 69, 9–15.

Ma, J., 1994. Strategy-proofness and the strict core in a market with indivisibilities. Int J Game Theory 23, 75–83.

Mishra, D., Parkes, D.C., 2009. Multi-item Vickrey-Dutch auctions. Games Econ Behavior 66, 326–347.

Miyagawa, E., 2001. House allocation with transfers. J Econ Theory 100, 329–355.

Miyagawa, E., 2002. Strategy-proofness and the core in house allocation problems. Games Econ Behav 38, 347–361.

Morimoto, S., Serizawa, S., 2014. Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule. Theoretical Econ, forthcoming.

Postlewaite, A., Roth A.E., 1977. Weak versus strong domination in a market with indivisible goods. J Math Econ 4, 131–137.

Quinzii, M., 1984. Core and competitive equilibria with indivisibilities. Int J Game Theory 13, 41–60.

Roth, A.E., 1982. Incentive compatibility in a market with indivisible goods. Econ Letters 9, 127–132.

Satterthwaite, M.A., Sonnenschein, H., 1981. Strategy-proof allocation mechanisms at differentiable points. Rev Econ Stud 48, 587–597.

Schummer, J., 2000. Eliciting preferences to assign positions and compensation. Games Econ Behav 30, 293Ű-318.

Shapley, L.S., Scarf, H., 1974. On cores and indivisibility. J Math Econ 1, 23–28.

Shapley, L.S., Shubik, M., 1971. The assignment game I: The core. Int J Game Theory 1, 111–130.

Sun, N., Yang, Z. 2003. A general strategy proof fair allocation mechanism. Econ Letters 81, 73–79.

Svensson, L-G., 1983. Large indivisibles: Analysis with respect to price equilibrium and fairness. Econometrica 51, 939–954.

Tadenuma, K., Thomson, W., 1991. No-envy and consistency in economies with indivisible goods. Econometrica 59, 1755–1767.

Vickrey, W., 1961. Counterspeculation, auctions, and competitive sealed tenders. J Finance 16, 8–37.

Discussion Papers of the Research Area Markets and Choice 2015

Research Unit: **Market Behavior**

Sebastian Kodritsch   SP II 2015–201
A note on the welfare of a sophisticated
time–inconsistent decision-maker

John Duffy, Dietmar Fehr   SP II 2015–202
Equilibrium selection in similar repeated games:
Experimental evidence on the role of precedents

Onur Kesten, Morimitsu Kurino, Alexander Nesterov   SP II 2015–203
Efficient lottery design

Dietmar Fehr, Rustamdjan Hakimov, Dorothea Kübler   SP II 2015–204
The willingness to pay-willingness to accept gap:
A failed replication of Plott and Zeiler

Dorothea Kübler, Julia Schmid   SP II 2015–205
Take your time to grow: A field experiment on the hiring of youths in
Germany

Julia Schmid   SPII 2015–206
Voluntary industry standards: An experimental investigation of
a Greek gift

Tommy Andersson, Lars Ehlers, Lars–Gunnar Svensson   SPII 2015–207
Transferring ownership of public housing to existing tenants: A
mechanism design approach

Research Unit: **Economics of Change**

Nikolaus Schweizer, Nora Szech   SP II 2015–301
Revenues and welfare in auctions with information release

Maja Adena, Steffen Huck   SP II 2015–302
Matching donations without crowding out? Some theoretical
considerations and a field experiment

Justin Valasek   SP II 2015–303
Reforming an institutional culture of corruption: A model of motivated
agents and collective reputation

Steffen Huck, Nora Szech, Lukas M. Wenner   SP II 2015–304
More effort with less pay: On information avoidance, belief design and
performance

All discussion papers are downloadable:
http://www.wzb.eu/en/publications/discussion-papers/markets-and-choice

WZB Junior Research Group: **Risk and Development**

Ferdinand M. Vieider, Clara Villegas–Palacio, Peter Martinsson,   SP II 2015–401
Milagros Mejía
Risk taking for oneself and others: A structural model approach