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Abstract

A Note on the Welfare of a Sophisticated Time-Inconsistent Decision-Maker

by Sebastian Kodritsch*

I examine the circumstances under which a sophisticated time-inconsistent decision-maker (i) will not or (ii) need not severely miscoordinate her behavior across time, in the sense of following a course of action which fails to be Pareto-optimal for the sequence of temporal selves of the individual (Laibson [1994] and O'Donoghue and Rabin [1999] provide prominent instances of such miscoordination). Studying the standard solution concept for this case—Strotz-Pollak equilibrium—in general decision problems with perfect information, I establish two results: first, for finite-horizon problems without indifference, essential consistency (Hammond [1976]) is sufficient for choice to be Pareto-optimal. Second, if the decision problem satisfies a certain history-independence property, whenever an equilibrium outcome fails to be Pareto-optimal, it is Pareto-dominated by another equilibrium outcome, leading to an existence result for a Pareto-optimal solution.

Keywords: time-inconsistency, multi-selves approach, Strotz-Pollak equilibrium, welfare, Pareto-optimality

JEL classification: C72, D11, D60, D74, D90

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1 Introduction

Strotz-Pollak equilibrium (StPoE) is the standard solution concept for intertemporal decision problems of individuals who have time-inconsistent preferences and perfectly know themselves. Dating back to the pioneering work of Strotz [1955-1956], this solution has been interpreted as the outcome of "consistent planning". Yet, a recurrent finding in applications is that outcomes thus obtained are inefficient according to the welfare criterion of Pareto-optimality when applied to the sequence of temporal selves of the decision maker; two well-known examples study the choices of (β, δ)-discounters in a timing problem (O'Donoghue and Rabin [1999, Proposition 5]) and in a consumption-savings problem (Phelps and Pollak [1968] or Laibson [1994, Chapter 1]), respectively. Such inefficient solutions represent instances of severe miscoordination of behaviour across time, which raises the question of what forms of dynamic inconsistency of preferences and environments permit or prevent this phenomenon.

This note presents welfare results about StPoE paths in general decision problems with perfect information. A main challenge in relating welfare rankings to equilibrium in general is the history-dependence of constraints as well as welfare. Nonetheless, allowing for arbitrary such history-dependence, the first result, proposition 1, provides a sufficient condition for "intrapersonal Pareto-optimality" of a StPoE path in finite-horizon problems without indifference: a limited form of intertemporal consistency of preferences, called "essential consistency" in reference to Hammond [1976] who originally advanced it, ensures this efficiency property. This result is illustrated and discussed with several examples of timing problems of a (β, δ)-discounter based on O'Donoghue and Rabin [1999].

Restricting the history-dependence inherent in the decision problem, corollary 1, relates welfare and multiplicity of StPoE paths by showing that under these restrictions, whenever a path supported by a StPoE is not intrapersonally Pareto-optimal, then any path that intrapersonally Pareto-dominates it, can also be supported by some StPoE. The welfare-rankability of multiple StPoE paths features prominently in various examples used to motivate refinements of StPoE—see Asheim [1997] and Kocherlakota [1996]—to which the result presented here adds a general observation. Together with proposition 2, which it is based upon, it also illuminates the occurrence of this phenomenon for the case of the consumption-savings problem of a (β, δ)-discounter introduced by Phelps and Pollak [1968] and rigorously analysed by Laibson [1994, Chapter 1].

2 Decision Problem

This section defines a general class of decision problems by a single decision-maker (DM) and the welfare criterion used throughout, and it presents two influential models from the literature which provide the running examples of this note.

2.1 Stages, Actions and Histories

There is a set of consecutive decision times $\mathcal{T} = \{t \in \mathbb{N}_0 \mid t < T\}$, where $T \in \mathbb{N}_0 \cup \{\infty\}$, at each of which a single DM takes an action a_t out of some non-empty, but possibly trivial (singleton), subset of a universal action space $\mathcal{A}$.[1] For any $t \in \mathcal{T}$, the set of all histories to time $t+1$ is denoted H^{t+1} and defined inductively from H^t via a mapping $A_t : H^t \to \mathcal{A}$, capturing constraints on actions at time t that evolve as a function of past choices: $H^0 \equiv \{\alpha\}$, where α is a parameter of the problem, and, for any $t \in \mathcal{T}$,

$$H^{t+1} = \left\{(h, a) \in H^t \times \mathcal{A} | a \in A_t\left(h\right)\right\}.$$

The set of terminal histories, called "paths", is then $H^T \equiv \Omega$, and the set of non-terminal histories, or—in what follows—simply "histories", is $\cup_{t \in \mathcal{T}} H^t \equiv \mathcal{H}$. It will be notationally convenient to also define a function $\tau : \mathcal{H} \cup \Omega \to \mathcal{T} \cup T$, such that, for any history $h \in \mathcal{H}$, $\tau\left(h\right) = t$ where $h \in H^t$, and, for any $\omega \in \Omega$, $\tau\left(\omega\right) = T$.

Generalising the above, for any $h \in \mathcal{H}$ and any time $t \geq \tau\left(h\right)$, define the set of histories to time t which are feasible after h, the "time-t continuations of h", denoted H^t_h, as follows: $H^{\tau(h)}_h \equiv \{h\}$ and, for any $t \geq \tau\left(h\right)$,

$$H^{t+1}_h = \left\{(h', a) \in H^t_h \times \mathcal{A} | a \in A_t\left(h'\right)\right\}.$$

Accordingly, the set of paths feasible after h is $H^T_h \equiv \Omega_h$, and the set of histories feasible after h is $\cup_{t \geq \tau(h)} H^t_h \equiv \mathcal{H}_h$.

Finally, define the mapping $\eta : \left(\mathcal{H} \cup \Omega\right)^2 \to \left(\mathcal{H} \cup \Omega\right)$ to associate with any pairwise combination of histories or paths the longest history such that both are feasible: for

[1] The possibility of trivial action spaces at various dates allows to capture discrete-time problems where decision dates are not equidistant in time, or also problems where after some time no decisions are to be made any more, while there are still welfare effects.

any $(x,y) \in (\mathcal{H} \cup \Omega)^2$,

$$\eta(x,y) = \begin{cases} x & x = y \\ h & x \neq y, \{x,y\} \subseteq \mathcal{H}_h \cup \Omega_h, \left[\forall a \in A_{\tau(h)}(h), \{x,y\} \nsubseteq \mathcal{H}_{(h,a)} \cup \Omega_{(h,a)}\right] \end{cases}.$$

Note that this is well-defined, because whenever $x \neq y$, there is a unique history h with the required property; moreover, $\eta(x,y) = \eta(y,x)$. For any two histories h and h', whenever $\eta(h,h') = h$, then say h is a subhistory of h' and h' is a continuation history of h; and for any history h and path ω, if $\eta(h,\omega) = h$, then call h a history along ω.

2.2 (Pure) Strategies

A pure strategy of the DM is a function $s : \mathcal{H} \to \mathcal{A}$ with the property that, for any $h \in \mathcal{H}$, $s(h) \in A_{\tau(h)}(h)$; let $\mathcal{S}$ denote the set of such functions. For any $h \in \mathcal{H}$ and any time $t \geq \tau(h)$, define a mapping $\omega_h^t : \mathcal{S} \to H^t$ inductively as follows, where $\omega_h^{\tau(h)}(s) \equiv h$, and

$$\omega_h^{t+1}(s) = \left(\omega_h^t(s), s\left(\omega_h^t(s)\right)\right).$$

Then, for any $h \in \mathcal{H}$, $s \in \mathcal{S}$ and date $t \geq \tau(h)$, $\omega_h^t(s)$ is the time-t continuation of h which results from following strategy s. Define $\omega_\alpha^t \equiv \omega^t$, so, in particular, $\omega^T(s)$ is simply the path under s.

For any $s \in \mathcal{S}$ and any $h \in \mathcal{H}$, denote the restriction of s to $\mathcal{H}_h$ by s_h and let $\mathcal{S}_h$ denote the set of functions thus obtained; elements of $\mathcal{S}_h$ will be called continuation strategies from h.

2.3 Preferences and Welfare Comparisons

At any time $t \in \mathcal{T}$, the DM has "preferences" over Ω which are represented by a function $U_t : \Omega \to \mathbb{R}$; note that, given domain Ω, U_t is allowed to vary with the particular history $h \in H^t$. Importantly, U_t goes beyond a representation of preferences in the usual sense: since it is defined for all paths at any time, two paths $\{\omega, \omega'\}$ may be compared even though there is no time-t history upon which both are actually feasible (formally, there does not exist any $h \in H^t$ such that $\{\omega, \omega'\} \subseteq \Omega_h$). Hence there is no choice experiment, not even under options with full commitment, that could elicit these "preferences". Thus U_t in fact measures the DM's welfare at time t for any path, and when feasible paths are compared, this implies preferences.

The welfare criterion I use throughout is a mere translation of the standard economic concept of Pareto efficiency into the language of dynamic paths and a single DM.

Definition 1. For any two paths $\{\omega, \omega'\} \subseteq \Omega$, ω *intrapersonally Pareto-dominates* (IP-dominates) ω' if, for any time $t \in \mathcal{T}$, $U_t(\omega) \geq U_t(\omega')$, and for some time $t' \in \mathcal{T}$, $U_{t'}(\omega) > U_{t'}(\omega')$. A path $\omega \in \Omega$ is *intrapersonally Pareto-optimal* (IP-optimal, "efficient") if there is no path $\omega' \in \Omega$ that IP-dominates ω.

2.4 Subproblems and Conventions

Denote any such decision problem by Γ; clearly, any history $h \in \mathcal{H}$ defines a decision problem of its own: by simply replacing h with α and times $t \geq \tau(h)$ with $t - \tau(h)$, it fits all the definitions above, and I will therefore denote this "subproblem" by $\Gamma(h)$. To simplify some of the notation here and in what follows, I make the convention that, for any history $h \in \mathcal{H}$ and any $t \in \mathcal{T}$, $\left(h, (a_s)_{s=t}^{t-1}\right) = h$. Moreover, when writing a history to some time t in explicit form as $(a_s)_{s=0}^{t-1}$, I usually omit α; however, $(a_s)_{s=0}^{-1} \equiv \alpha$.

2.5 Examples

This work focuses attention on two examples, which are among the most influential contributions to the analysis of decision making with time-inconsistent preferences. The first one is the model of O'Donoghue and Rabin [1999]: a (β, δ)-discounter chooses when to engage in a one-time activity before a deadline, where the activity yields immediate and delayed rewards as well as costs that vary with the timing of the activity. Real-life applications include the choice of when to prepare a report, visit a doctor for a medical check-up or go on a vacation.

Example 1. Let the "deadline" be $T < \infty$, set $\alpha = 0$ and $\mathcal{A} = \{0, 1\}$, where, for any $t \in \mathcal{T}$ and $h = \left(\alpha, (a_s)_{s=0}^{t-1}\right) \in H^t$, $z_t(h) \equiv \max\{a_s\}_{s=0}^{t-1}$ and $A_t(h) \equiv \{0, 1 - z_t(h)\}$. Action $a = 1$ at time t, when available, means that the DM performs the activity in period t; $A_t(h) = \{0\}$ if she has performed it in the past, though there still are welfare consequences to consider. The set of paths can be characterised by the timing of the activity: $\Omega = \mathcal{T} \cup T$, where $\omega = T$ is interpreted as performing the activity right at the deadline, when it must be done.[2] Let there be two non-negative functions $v : \Omega \to \mathbb{R}_+$

[2] For example, ignoring the initial history, if $T = 3$, then $\omega = 1$ is the path $(0, 1, 0)$ and $\omega = 3$ is the path $(0, 0, 0)$. See the discussion in O'Donoghue and Rabin [1999, p. 107, in particular footnote 12].

and $c : \Omega \to \mathbb{R}_+$, which define welfare together with a parameter β such that $0 < \beta \leq 1$, and distinguish two different types of problem.[3] First, a problem with immediate costs (and delayed rewards) is one where:

$$U_t(\omega) = \begin{cases} \beta\left(v(\omega) - c(\omega)\right) & t < \omega \\ \beta v(\omega) - c(\omega) & t = \omega \\ \beta v(\omega) & t > \omega \end{cases}.$$

The other type of this problem has immediate rewards (and delayed costs) instead:

$$U_t(\omega) = \begin{cases} \beta\left(v(\omega) - c(\omega)\right) & t < \omega \\ v(\omega) - \beta c(\omega) & t = \omega \\ -\beta c(\omega) & t > \omega \end{cases}.$$

Given how Ω is defined, the reward- and cost-schedules can be written as vectors of length $T + 1$, so I will use the notation $v = (v_t)_{t=0}^{T}$ and $c = (c_t)_{t=0}^{T}$, where v_t and c_t are the reward- and cost-values, respectively, when the activity is performed in period t.

The second example is based on the formulation of Plan [2010, Example 4] of the following problem originally proposed by Phelps and Pollak [1968] and reinterpreted as well as further analysed by Laibson [1994, Chapter 1]: a (β, δ)-discounter with constant relative risk aversion facing a constant return on savings chooses a discrete consumption-savings path over an infinite time-horizon.[4]

Example 2. Let $T = \infty$, $\alpha = W_0 > 0$ and, for any $t \in \mathcal{T}$, $A_t = A = [0, 1]$. W_0 is the DM's initial wealth and $a \in A$ is the fraction of wealth saved for the future in any period. With a constant gross interest rate of $R \geq 0$ and a given history $h = \left(W_0, (a_s)_{s=0}^{t-1}\right)$ to time $t \in \mathcal{T}$, wealth at time t equals $W_t = R^t \left(\prod_{s=0}^{t-1} a_s\right) W_0$. Preferences, and in fact welfare, are parameterised by (β, δ, ρ) with $0 < \beta \leq 1$, $0 < \delta < 1$ and $\rho < 1$, where the

[3] The assumption about the (β, δ)-discounter that $\delta = 1$ is immaterial; see O'Donoghue and Rabin [1999, footnote 11] which shows that any "long-term discounting" can be incorporated in v and c.

[4] See also Barro [1999] for a variant of this problem in continuous time with more general time-varying time preferences and a neoclassical production technology, Krusell and Smith-Jr. [2003] who investigate stationary savings rules for more general (instantaneous) utility functions and savings technologies, or Bernheim et al. [2013] who extend this problem to the case of a credit constraint (a lower bound on assets at any time).

standard restriction that $\delta R^{1-\rho}<1$ is imposed:

$$\begin{aligned}
U_t\left(W_0,(a_s)_{s=0}^{\infty}\right) &= \left((1-a_t)\,W_t\right)^{1-\rho}+\beta\sum_{s=t+1}^{\infty}\delta^{s-t}\left((1-a_s)\,R^{s-t}\left(\prod_{r=t}^{s-1}a_t\right)W_t\right)^{1-\rho}\\
&= W_t^{1-\rho}\underbrace{\left((1-a_t)^{1-\rho}+\beta\sum_{s=t+1}^{\infty}\delta^{s-t}\left((1-a_s)\,R^{s-t}\left(\prod_{r=t}^{s-1}a_t\right)\right)^{1-\rho}\right)}_{\equiv U\left((a_s)_{s=t}^{\infty}\right)}\\
&= \left(R^t\left(\prod_{s=0}^{t-1}a_s\right)W_0\right)^{1-\rho}U\left((a_s)_{s=t}^{\infty}\right).
\end{aligned}$$

Note that this decision problem satisfies a history-independence property (see definition 5 below): action sets are constant and history enters welfare in a multiplicative manner, which means it does not affect the ranking of feasible continuation plans; the latter is always represented by the function $U:[0,1]^T\to\mathbb{R}$ as defined above.[5]

3 Choice and Welfare

3.1 Strotz-Pollak Equilibrium

Strotz [1955-1956] pioneered the analysis of a time-inconsistent DM's behaviour in the context of a deterministic continuous-time consumption problem. He suggested that a DM who correctly anticipates her future preferences, a "sophisticated" DM, would select "the best plan among those that he will actually follow" (Strotz [1955-1956, p. 173]), which Pollak [1968, Section 1] formalised for a discretised version of the original problem. Early generalisations of this definition can be found in Peleg and Yaari [1973, p. 395], Goldman [1979], pointing out the equivalence with (a particular application of) subgame-perfect Nash equilibrium (SPNE), and Goldman [1980], where the terminology of "Strotz-Pollak equilibrium" that the literature has adopted is introduced. Laibson [1994] describes the general solution as the SPNE of the "intrapersonal game" where each temporal self of the DM is defined to be a distinct non-cooperative player. The same approach has been applied to decision problems featuring imperfect recall (see

[5] Phelps and Pollak [1968] and Laibson [1994, Chapter 1] formulate this problem with absolute consumption as the action chosen in any period, subject to the wealth constraint, which is history-dependent.

Piccione and Rubinstein [1997] and other contributions to the same (special) journal issue).

Definition 2. A strategy $\hat{s} \in \mathcal{S}$ is a *Strotz-Pollak equilibrium* (StPoE) if, for any $h \in \mathcal{H}$ and $a \in A_{\tau(h)}(h)$,

$$U_{\tau(h)}\left(\omega_h^T(\hat{s})\right) \geq U_{\tau(h)}\left(\omega_{(h,a)}^T(\hat{s})\right).$$

A path $\hat{\omega} \in \Omega$ is a *Strotz-Pollak solution* (StPo-solution) if there exists a StPoE $\hat{s} \in \mathcal{S}$ such that $\omega^T(\hat{s}) = \hat{\omega}$.

StPoE requires that, at any history h, the DM best-responds to correct beliefs about future behaviour such that this behaviour, at any future history, is a best response to the same beliefs. The DM cannot commit to future actions but forms beliefs about them which, when shared at all histories, imply rational behaviour. As is clear from the definition as well as this description, if $\hat{s}$ is a StPoE of $\Gamma(\alpha)$, then, for any history $h \in \mathcal{H}$, $\hat{s}_h$ is a StPoE of $\Gamma(h)$ (the converse holds true as well, of course).

StPoE is an application of SPNE to the game with the same extensive form, but where a separate non-cooperative player acts at each decision time (equivalently, at each history, because only one history can be played to any given decision time). Thus, well-known existence theorems for SPNE apply, e.g. Harris [1985].[6] It shares the notion of "credibility" inherent in SPNE, where, fixing beliefs, the DM does not expect to take actions in the future that she would not find optimal once the contingency were to actually occur. Applied to a single DM with perfect self-knowledge, this could be termed loosely as ruling out that she "fool" herself.

3.2 Essential Consistency and Welfare

Recall example 1 with immediate rewards for $T = 2$, where $\beta = \frac{1}{2}$ and the reward- and cost-schedules are given by

$$\begin{aligned} v &= (0, 5, 1) \\ c &= (1, 8, 0). \end{aligned}$$

This results in the following unique StPoE: since $U_1(1) = 5 - \frac{1}{2}8 > \frac{1}{2}1 = U_1(2)$, the DM in period 1 would engage in the activity. Therefore, it will actually be performed imme-

[6] See also Goldman [1980] who establishes existence of StPoE in a general class of finite-horizon problems, where Peleg and Yaari [1973] had initially raised concerns about non-existence despite "well-behaved" settings.

diately: $U_0(0) = -\frac{1}{2}1 > -\frac{1}{2}(5-8) = U_0(1)$. Compare now the welfare consequences from this outcome to that if the DM waited until period 2 instead: $U_0(0) = U_1(0) = -\frac{1}{2}$, whereas $U_0(2) = U_1(2) = \frac{1}{2}$. The unique StPo-solution is therefore IP-dominated. The reason the DM does not wait initially, even though she would strictly prefer doing it in period 2 rather than now, is that she would otherwise do it next period; at that point, however, she would prefer the (then) immediate reward over waiting yet another period.

Clearly, these preferences are time-inconsistent, because the DM's preferences as of the initial period over doing it next period and doing it the period after that reverse once the next period arrives. In order to make terms precise, I provide a definition of the benchmark of time-consistency here.

Definition 3. (TC) Preferences are *time-consistent* if, for any history $h \in \mathcal{H}$ and any two paths $\{\omega, \omega'\} \subseteq \Omega_h$ (feasible after h),

$$U_{\tau(h)}(\omega) \geq U_{\tau(h)}(\omega') \Leftrightarrow U_0(\omega) \geq U_0(\omega').$$

When preferences are time-consistent, there is a single utility function—without loss of generality, it is chosen to be U_0—that represents the DM's preferences over *feasible* paths at any history. Accordingly, if a path is optimal for the DM at the initial date 0, then it remains optimal for the DM at any history along that path; in particular, if, at the outset of the problem, the DM has a unique optimal path, then it remains uniquely optimal for the DM at any history along this path among all the paths feasible at that history.

In the example above, however, the nature of the violation of time-inconsistency is special. Notice the following intertemporal cycle: at $t = 1$, the DM prefers doing it in period 1 over doing it in period 2, whereas at $t = 0$, the DM prefers doing it in period 2 over doing it in period 0 which is in turn preferred to doing it in period 1. This constitutes a violation of the following property.[7]

Definition 4. (EC) Preferences are *essentially consistent* if, for any pair of histories $\{h, h'\} \subseteq \mathcal{H}$ and triple of paths $\{\omega, \omega', \omega''\} \subseteq \Omega$ such that $h = \eta(\omega, \omega') = \eta(\omega, \omega'')$ and $h' = \eta(\omega', \omega'') \in \mathcal{H}_h$,

$$U_{\tau(h')}(\omega') > U_{\tau(h')}(\omega'') \wedge U_{\tau(h)}(\omega) > U_{\tau(h)}(\omega') \quad \Rightarrow \quad U_{\tau(h)}(\omega) > U_{\tau(h)}(\omega''). \qquad (1)$$

[7] In the definition's notation, $\omega = 0$, $\omega' = 1$ and $\omega'' = 2$; these are compared as of $t = 0$ and $t = 1$.

I formulate this consistency property in strict terms because I only use it in proposition 1 which rules out indifference. Moreover, it is thus identical to the property advanced by Hammond [1976], who showed—again for the case of no indifference—that essential consistency ensures the coincidence of sophisticated and naïve choices in finite decision trees ($T < \infty$ and $\mathcal{A}$ finite). It requires that sophisticated choice from $\{\omega, \omega', \omega''\}$ at history h not change when an alternative that is not chosen but still available at future history h' is removed.

Clearly, essential consistency is implied by time-consistency; however, it is indeed weaker: assuming no indifference and considering the same paths and histories as in the definition, when $U_{\tau(h)}(\omega) < \min\left\{U_{\tau(h)}(\omega'), U_{\tau(h)}(\omega'')\right\}$, it does not restrict preferences at history h over $\{\omega', \omega''\}$, nor when both $U_{\tau(h)}(\omega) > U_{\tau(h)}(\omega')$ and $U_{\tau(h')}(\omega') < U_{\tau(h')}(\omega'')$ are true, whereas time-consistency requires they coincide with those at history h'.

Remark 1. If preferences are time-consistent, then they are essentially consistent, but the converse is not true.

Proof. Suppose TC and consider histories and paths as in the definition of EC. Under TC, the antecedent in (1) is equivalent to

$$U_0(\omega') > U_0(\omega'') \wedge U_0(\omega) > U_0(\omega'),$$

which, by transitivity of $>$, yields $U_0(\omega) > U_0(\omega'')$, and applying TC once more gives $U_{\tau(h)}(\omega) > U_{\tau(h)}(\omega'')$.

For a counterexample to the converse, consider example 1 for $T = 2$ with immediate costs and $\beta = \frac{1}{2}$, where $v = (3, 3, 1)$ and $c = (2, 2, 1)$, so

$$\begin{array}{c} U_0(0) = -\frac{1}{2} < U_0(2) = 0 < U_0(1) = \frac{1}{2} \\ U_1(1) = -\frac{1}{2} < U_1(2) = 0 \end{array},$$

so these clearly violate TC. In contrast, EC is satisfied because—in the definition's notation—it must be that $\omega = 0$, whence the antecedent of (1) is vacuous here. □

In finite-horizon settings without any indifference essential consistency guarantees that the StPo-solution—there is a unique one by backwards induction—is IP-optimal. Alternatively put, if a StPo-solution is found to be inefficient by the Pareto-criterion, it must be that preferences violate essential consistency.[8] The proof of this result uses the following lemma, which exploits the structure of StPoE based on backwards induction when the horizon is finite.

Lemma 1. *Let $T < \infty$, suppose $\hat{\omega}$ is a StPo-solution and take any other path $\omega = (a_t)_{t=0}^{T-1} \in \Omega \setminus \{\hat{\omega}\}$. Then there exist an integer K with $0 < K \leq T$ and a sequence of paths $(\omega_k)_{k=0}^{K}$ with $\omega_0 = \hat{\omega}$ and $\omega_K = \omega$ such that, for any $k \in \{0, \ldots, K-1\}$,*

$$\begin{aligned} \eta(\omega_{k+1}, \omega) &\in \mathcal{H}_{\eta(\omega_k,\omega)} \setminus \{\eta(\omega_k, \omega)\} \\ U_{\tau(\eta(\omega_k,\omega))}(\omega_{k+1}) &\leq U_{\tau(\eta(\omega_k,\omega))}(\omega_k). \end{aligned}$$

Proof. Let $\hat{s}$ be a StPoE such that $\hat{\omega} = \omega^T(\hat{s})$ and construct a sequence of paths $(\omega_0, \omega_1, \ldots)$ as follows: set $h_0 \equiv \alpha$, and iterate

$$\begin{aligned} \omega_k &\equiv \omega_{h_k}^T(\hat{s}) \\ h_{k+1} &\equiv \left(\eta(\omega_k, \omega), a_{\tau(\eta(\omega_k,\omega))}\right) \end{aligned}$$

until $\omega_k = \omega$, in which case set $K = k$. It is easily checked that this sequence satisfies $0 < K \leq T$, $\omega_0 = \hat{\omega}$ and $\eta(\omega_{k+1}, \omega) \in \mathcal{H}_{\eta(\omega_k,\omega)} \setminus \{\eta(\omega_k, \omega)\}$.

Denote, for simplicity, $t_k \equiv \tau(\eta(\omega_k, \omega))$ and suppose now there is a $k \in \{0, \ldots, K-1\}$ such that $U_{t_k}(\omega_k) < U_{t_k}(\omega_{k+1})$. Letting $h = \eta(\omega_k, \omega)$, this would therefore imply that

$$U_{\tau(h)}\left(\omega_h^T(\hat{s})\right) < U_{\tau(h)}\left(\omega_{\left(h, a_{\tau(h)}\right)}^T(\hat{s})\right),$$

which contradicts that $\hat{s}$ is a StPoE. □

Proposition 1. *Let $T < \infty$ and assume preferences exhibit no indifference in the sense that, for any time $t \in \mathcal{T}$ and any two paths $\{\omega, \omega'\} \subseteq \Omega_h$ (feasible after h) with $\omega \neq \omega'$, $U_t(\omega) \neq U_t(\omega')$ holds true. Then there is a unique StPo-solution, and if preferences are essentially consistent, it is IP-optimal.*

[8] While IP-dominance, applied to example 1, compares present discounted utilities, O'Donoghue and Rabin [2001, Section V] demonstrate an even stronger "dominance" property: there exists another performance period which yields *instantaneous* utility at least as great in every period and greater in some period than the unique StPo-solution (with the above numbers, the respective sequences of instantaneous utilities for periods 0, 1 and 2 are $(0, 0, 0)$ for the StPo-solution and $(0, 0, 1)$ for performance in period 2 instead). Of course, their criterion is applicable only in discounted-utility models.

Proof. Uniqueness of StPoE in this finite-horizon problem follows from backwards induction, since there is no indifference. Denote this unique StPoE by $\hat{s}$ and the associated unique StPo-solution by $\hat{\omega}$.

Take any path $\omega \neq \hat{\omega}$ and consider a sequence $(\omega_k)_{k=0}^{K}$ as in lemma 1; because there is no indifference, for any $k \in \{0, \ldots, K-1\}$, $U_{t_k}(\omega_k) > U_{t_k}(\omega_{k+1})$. In particular, $U_{t_{K-1}}(\omega_{K-1}) > U_{t_{K-1}}(\omega)$, and if $K = 1$, then $\omega_{K-1} = \hat{\omega}$, whence ω does not IP-dominate $\hat{\omega}$. If $K > 1$, since for any $k' > k$, $\omega_{k'} \in \Omega_{h_k}$, we can apply EC as follows:

$$\begin{array}{rcl} U_{t_{K-1}}(\omega_{K-1}) > U_{t_{K-1}}(\omega) & \wedge & U_{t_{K-2}}(\omega_{K-2}) > U_{t_{K-2}}(\omega_{K-1}) \\ & \Rightarrow & \\ U_{t_{K-2}}(\omega_{K-2}) & > & U_{t_{K-2}}(\omega). \end{array}$$

If $K = 2$ then $\omega_{K-2} = \hat{\omega}$, so this means ω does not IP-dominate $\hat{\omega}$. If $K > 2$ then apply EC once more:

$$\begin{array}{rcl} U_{t_{K-2}}(\omega_{K-2}) > U_{t_{K-2}}(\omega) & \wedge & U_{t_{K-3}}(\omega_{K-3}) > U_{t_{K-3}}(\omega_{K-2}) \\ & \Rightarrow & \\ U_{t_{K-3}}(\omega_{K-3}) & > & U_{t_{K-3}}(\omega). \end{array}$$

If $K = 3$ then $\omega_{K-3} = \hat{\omega}$, so this means ω does not IP-dominate $\hat{\omega}$. Since $K < \infty$, applying EC $K-1$ times, this process will eventually yield $U_{t_0}(\hat{\omega}) > U_{t_0}(\omega)$, implying that ω does not IP-dominate $\hat{\omega}$. This is true for any $\omega \neq \hat{\omega}$, whence $\hat{\omega}$ is IP-optimal. □

Discussion

Essential consistency rules out intertemporal cycles: when later the DM will be decisive about ω' versus ω'' in favour of ω' and now decides about $\{\omega\}$ versus $\{\omega', \omega''\}$, she does not prefer ω'' to ω and ω to ω'. The proof of proposition 1 shows that, for each alternative path that is not the unique StPo solution, there exists a time $t \in \mathcal{T}$ at which the DM prefers the solution to that path; in fact, this t is the first time the DM's action deviates from the alternative path. Considering the generality of the decision problem in terms of the history-dependence of welfare, this is a remarkable result, despite the strength of essential consistency.

For an illustration of this efficiency result when preferences are time-inconsistent, recall the special case of example 1 used in remark 1, where it was established that preferences are indeed essentially consistent. The unique StPo-solution is to wait until period 2 to perform the task, and this is IP-optimal: the time-0 DM prefers this path to doing it immediately, and the same is true at time 1 about doing it immediately then instead.

In a special case of example 1, essential consistency is also necessary.

Remark 2. In example 1 with $T = 2$, $\beta < 1$ and immediate rewards, the unique StPo-solution is IP-optimal if and only if preferences are essentially consistent.

Proof. Sufficiency follows from proposition 1, so suppose EC were violated. This means either (i) $U_1(1) > U_1(2)$ and $U_0(2) > U_0(0) > U_0(1)$ or (ii) $U_1(2) > U_1(1)$ and $U_0(1) > U_0(0) > U_0(2)$. However, (ii) cannot hold with immediate rewards because:

$$\begin{aligned} U_1(2) > U_1(1) &\Leftrightarrow \beta(v_2 - c_2) > v_1 - \beta c_1 \\ U_0(1) > U_0(2) &\Leftrightarrow \beta(v_1 - c_1) > \beta(v_2 - c_2), \end{aligned}$$

which implies $\beta v_1 > v_1$, a contradiction (since $\beta < 1$ and $v_1 \geq 0$).

Consider then case (i): the unique StPo-solution is that the activity is performed immediately. This path is IP-dominated by waiting to do it in period 2 whenever $U_1(2) > U_1(0)$, i.e. $\beta(v_2 - c_2) > -\beta c_0$; the latter is, however, an implication of $U_0(2) > U_0(0)$ because $v_0 \geq 0$:

$$U_0(2) > U_0(0) \Leftrightarrow \beta(v_2 - c_2) > v_0 - \beta c_0.$$

□

For longer horizons, an essential inconsistency may be irrelevant to the StPo-solution. Informally, if one added a new initial period in which the DM prefers doing it immediately over any other outcome, this would result in an IP-optimal StPo-solution, irrespective of whether in the subproblem after waiting initially there is an essential inconsistency or not. Hence, essential consistency certainly needs to be weakened further for a characterisation of IP-optimality in example 1 with immediate rewards when $T > 2$, or even beyond to cope with both immediate rewards and immediate costs.

Significantly generalising proposition 1 to dealing with indifference would require first a modification of the notion of essential consistency and hardly appears promising.[9]

[9] One conclusion is immediate from lemma 1, however, when in the above definition of essential consistency (1) is instead formulated with weak preferences: no StPo-solution is strongly IP-dominated (IP-dominance with strict "preference" for every time t).

Again using example 1 with $T = 2$ and $\beta < 1$, note that whenever at $t = 1$ the DM is indifferent between the two remaining feasible paths, she has a strict preference at $t = 0$ (with immediate costs for doing it in period 1 and with immediate rewards for doing it in period 2). Depending on v_0 and c_0, how this indifference at $t = 1$ translates into (expected) choice at $t = 1$ may determine behaviour at $t = 0$ and consequently result in two different StPo-solutions where one IP-dominates the other in a manner orthogonal to essential consistency.

Relatedly, when moving toward an infinite horizon, the assumption of no indifference becomes hardly defensible. Moreover, essential consistency loses its force as sequences constructed on the basis of the proof of lemma 1 become infinite.[10] Indeed, the work of Laibson [1994, Chapter 1, Section 3] shows that this extends to the case of even time-consistent preferences when payoffs are unbounded from below in example 2 (time-consistency there means $\beta = 1$): letting $\rho > 1$, any path can be supported as StPo-solution by the threat that, upon any past deviation, consumption would take place at a (constant) rate sufficiently close to one (the continuation payoff approaches negative infinity). Even with bounded payoffs, Plan [2010, Footnote 12] shows how, with infinite cascades of threats of ever lower savings rates, one can construct a StPoE such that at every *history*, adhering to it makes the DM better off than the stationary, constant-savings-rate StPoE proposed by Phelps and Pollak [1968] (the latter features undersaving and is used as the limiting savings rate of the punishment cascade).

3.3 History-Independence, Welfare and Multiplicity

The previous section presented a sufficiency result for the IP-optimality of a StPo-solution, and its discussion indicated how this welfare property may fail more generally. Relatedly, an argument used to discard particular StPo-solutions is that they are IP-dominated *by other StPo-solutions*: this phenomenon is shared by most examples that the literature introducing refinements of StPoE has produced, e.g. Kocherlakota [1996] or Asheim [1997]. While hardly made explicit, the argument seems to be that it reflects an implausible failure of coordination in that the beliefs arrived at are self-defeating: there is another "credible" path that IP-dominates the one resulting from those beliefs, so a "planning" DM will never coordinate future beliefs on such a strategy.

This section addresses the question of when this form of Pareto-rankable multiplicity obtains and thus also provides insights into existence of an IP-optimal StPo-solution.

[10] See also the discussion of essential consistency in infinite trees by Hammond [1976, pp. 170-171].

In order to be able to do so, I restrict the history-dependence inherent in the decision problem. Recall that welfare at any time is defined for all paths, whence also paths that are never altogether feasible are compared by the welfare criterion (see section 2.3). In contrast, for equilibrium choices, only comparisons of feasible paths matter. Without restrictions on the nature of history-dependence, welfare comparisons of feasible paths may not provide any information about welfare at other paths, and it is impossible to uncover implications for equilibrium properties from the welfare criterion in general. Since, to the best of my knowledge, this work is the first investigation of welfare of StPo-solutions beyond particular models, I consider the following rather strong properties.

Definition 5. A decision problem satisfies *history-independence* if, for any time $t \in \mathcal{T}$ and any two histories $\{h, h'\} \subseteq H^t$, (i) $A_t(h) = A_t(h') \equiv A_t$, and, (ii), for any two sequences of continuation play $\left\{(a_s)_{s=t}^{T-1}, (a'_s)_{s=t}^{T-1}\right\} \subseteq \times_{s=t}^{T-1} A_s$,

$$U_t\left(h, (a_s)_{s=t}^{T-1}\right) \geq U_t\left(h, (a'_s)_{s=t}^{T-1}\right) \Leftrightarrow U_t\left(h', (a_s)_{s=t}^{T-1}\right) \geq U_t\left(h', (a'_s)_{s=t}^{T-1}\right).$$

It satisfies history-independence *even in a welfare sense* if (ii) is replaced by (ii*), for any continuation play $(a_s)_{s=t}^{T-1} \in \times_{s=t}^{T-1} A_s$, $U_t\left(h, (a_s)_{s=t}^{T-1}\right) = U_t\left(h', (a_s)_{s=t}^{T-1}\right)$.

History-independence of a decision problem means that, after any two histories to a particular date, the sets of feasible continuations are (i) identical (history-independent constraints) and (ii) ranked the same way (history-independent *preferences*).[11] It does not imply that welfare is unaffected by past choices, however, which is true only upon replacing (ii) with (ii*); clearly, the latter is stronger.[12] Example 2 illustrates this point, since initial wealth in any period (more precisely, a positive transformation of wealth), which is determined by past savings choices, enters the utility function multiplicatively, whereby it does not affect the rankings of continuation paths; thus (ii) holds whereas (ii*) is violated. This example also demonstrates that there are nonetheless important economic decision problems featuring dynamic constraints that (can be formulated so they) satisfy history-independence (see Plan [2010] for a closely related point).

The essence of history-independence is that, conditional on time, the DM's contin-

[11] Note that (ii) relies on (i) to be well-defined; although one could define the history-independence of preferences independently to hold only when continuation plays are actually feasible under both histories, for the purposes here, this is unnecessary as (ii) is only considered in problems which satisfy (i) anyways.

[12] (ii*) implies $U_t\left(h, (a_s)_{s=t}^{T-1}\right) = U_t\left(h', (a_s)_{s=t}^{T-1}\right)$ and $U_t\left(h, (a'_s)_{s=t}^{T-1}\right) = U_t\left(h', (a'_s)_{s=t}^{T-1}\right)$ from which (ii) follows.

uation behaviour can always ignore the past: any continuation play that is feasible at *some* history is feasible after *any* history, whence, if some continuation play constitutes a StPoE after that history, because of (ii) in definition 5, this is true after any other history to the same decision time; this is the content of the following lemma.

Lemma 2. *Assume the decision problem satisfies history-independence. Take a strategy $s \in \mathcal{S}$ such that s_h is a StPoE of $\Gamma(h)$ for history $h \in \mathcal{H}$ and consider any history $h' \in \mathcal{H}$ with $\tau(h') = \tau(h) = \tau$. Then any strategy $s' \in \mathcal{S}$ such that, for any non-negative integer $k \leq T - \tau$ and $(a_t)_{t=\tau}^{\tau+k-1} \in \times_{t=\tau}^{\tau+k-1} A_t$,*

$$s'\left(h', (a_t)_{t=\tau}^{\tau+k-1}\right) = s\left(h, (a_t)_{t=\tau}^{\tau+k-1}\right)$$

satisfies that $s'_{h'}$ is a StPoE of $\Gamma(h')$.

Proof. Suppose $s'_{h'}$ is not a StPoE of $\Gamma(h')$, so there exist a history $\hat{h}' \in \mathcal{H}_{h'}$ and an action $\bar{a} \in A_{\tau(\hat{h}')}$ such that

$$U_{\tau(\hat{h}')}\left(\omega_{(\hat{h}',\bar{a})}^{T}(s')\right) > U_{\tau(\hat{h}')}\left(\omega_{\hat{h}'}^{T}(s')\right).$$

Note that, by part (i) of history-independence of a decision problem as in definition 5, $\hat{h}' = \left(h', (a_t)_{t=\tau}^{\tau+k-1}\right)$ for some non-negative integer $k \leq T - \tau$, and consider $\hat{h} = \left(h, (a_t)_{t=\tau}^{\tau+k-1}\right)$. The definition of s' on $\mathcal{H}_{h'}$ via s on $\mathcal{H}_h$ implies that $\omega_{(\hat{h}',\bar{a})}^{T}(s')$ and $\omega_{(\hat{h},\bar{a})}^{T}(s)$ are identical from time τ onwards, and the same is true about the two paths $\omega_{\hat{h}'}^{T}(s')$ and $\omega_{\hat{h}}^{T}(s)$. Therefore, part (ii) of a decision problem's history-independence yields that

$$U_{\tau(\hat{h})}\left(\omega_{(\hat{h},\bar{a})}^{T}(s)\right) > U_{\tau(\hat{h})}\left(\omega_{\hat{h}}^{T}(s)\right).$$

This, however, contradicts the hypothesis that s_h is a StPoE of $\Gamma(h)$. □

Lemma 2 allows to establish a multiplicity result about history-independent decision problems, which is related to the welfare criterion in the discussion that follows, precisely in corollary 1 which uses definition 6.

Proposition 2. *Assume the decision problem satisfies history-independence and let $\hat{\omega} = (\hat{a}_t)_{t=0}^{T-1}$ be a StPo-solution; then any other path $\tilde{\omega} = (\tilde{a}_t)_{t=0}^{T-1} \neq \hat{\omega}$ such that, for any time $t \in \mathcal{T}$,*

$$U_t(\tilde{\omega}) \geq U_t\left((\tilde{a}_s)_{s=0}^{t-1}, (\hat{a}_s)_{s=t}^{T-1}\right), \tag{2}$$

is also a StPo-solution.

Proof. Take any StPoE $\hat{s}$ with $\omega^T(s) = \hat{\omega}$ and consider the strategy $\tilde{s}$ constructed as follows: whenever a history $h \in \mathcal{H}$ satisfies $\tilde{\omega} \in \Omega_h$, set $\tilde{s}(h) = \tilde{a}_{\tau(h)}$; then note that any other history can be written as $h = \left(h', \bar{a}, (a_t)_{t=\tau(h')+1}^{\tau(h')+k}\right)$ for some $k \in \mathbb{Z}$ with $0 \leq k \leq T - \tau(h') - 1$ and where $\eta(h, \tilde{\omega}) = h'$, in which case set

$$\tilde{s}(h) = \hat{s}\left((\hat{a}_t)_{t=0}^{\tau(h')-1}, \bar{a}, (a_t)_{t=\tau(h')+1}^{\tau(h')+k}\right).$$

This defines $\tilde{s}$ for every history h such that $\tilde{\omega} \notin \Omega_h$.

It will now be shown that $\tilde{s}$ is a StPoE and thus that $\tilde{\omega}$ is indeed a StPo-solution. Consider first any history h with $\tilde{\omega} \notin \Omega_h$ and note that there exist a history h' and an action $\bar{a} \in A_{\tau(h')}$ such that $\tilde{\omega} \in \Omega_{h'}$, $\tilde{\omega} \notin \Omega_{(h',\bar{a})}$ and $h \in \mathcal{H}_{(h',\bar{a})}$. Since, for $h'' = \left((\hat{a}_t)_{t=0}^{\tau(h')-1}, \bar{a}\right)$, $\hat{s}_{h''}$ is a StPoE of $\Gamma(h'')$, lemma 2 establishes that $\tilde{s}_{(h',\bar{a})}$ is a StPoE of $\Gamma(h', \bar{a})$; because $h \in H_{(h',\bar{a})}$, $\tilde{s}_h$ is therefore a StPoE of $\Gamma(h)$.

Now take a history h with $\tilde{\omega} \in \Omega_h$ and consider any $a \in A_{\tau(h)}$ with $a \neq \tilde{s}(h) = \tilde{a}_{\tau(h)}$. By definition of $\tilde{s}$, at all times $t \geq \tau(h)$, the actions on path $\omega_{(h,a)}^T(\tilde{s})$ are identical to those on path $\omega_{(h',a)}^T(\hat{s})$ when $h' = (\hat{a}_t)_{t=0}^{\tau(h)-1}$; using that $\tau(h') = \tau(h)$, since $\hat{s}$ is a StPoE, $U_{\tau(h)}\left(\omega_{(h',a)}^T(\hat{s})\right) \leq U_{\tau(h)}\left(\omega_{h'}^T(\hat{s})\right) = U_{\tau(h)}(\hat{\omega})$. The history-independence of preferences (property (ii) of definition 5) implies $U_{\tau(h)}\left(\omega_{(h,a)}^T(\tilde{s})\right) \leq U_{\tau(h)}\left(h, (\hat{a}_t)_{t=\tau(h)}^{T-1}\right)$. Combining this last inequality with $U_{\tau(h)}\left(h, (\hat{a}_t)_{t=\tau(h)}^{T-1}\right) \leq U_{\tau(h)}(\tilde{\omega})$, from the hypothesis of the proposition, one finally obtains $U_{\tau(h)}\left(\omega_{(h,a)}^T(\tilde{s})\right) \leq U_{\tau(h)}\left(\omega_h^T(\tilde{s})\right)$, completing the proof. □

Discussion

While it may appear that lemma 2 should immediately yield that if a path IP-dominates a StPo-solution, that path must be supportable by a StPoE as well—it could be based on the very same "threats"—this is not true in general. Consider the following simple example of a history-independent decision problem: $T = 2$, $A_0 = A_1 = \{0, 1\}$, $U_0(a_0, a_1) = -|a_0 - a_1|$ and $U_1(a_0, a_1) = 2a_0 - a_1$. At time $t = 1$, the DM prefers 0 over 1 irrespective of the previous action, whence she matches this action with $a_0 = 0$ at $t = 0$. Yet, this unique StPo-solution $(0, 0)$ is IP-dominated by $(1, 1)$. Note how this example relies on the history-dependence of *welfare* in the second period.

However, proposition 2 illuminates example 2: there is a unique StPoE with the property that consumption/saving takes place at the same rate irrespective of time and history. This "simple" equilibrium was first identified by Phelps and Pollak [1968, Part IV], who also showed that the resulting path is IP-dominated by other constant-rate paths of consumption/saving. Because we are comparing constant-rate paths, inequality 2 holds true: to see this, first note that when $a_s = \tilde{a} > 0$ for all $s \in \mathcal{T}$, then, for any $t \in \mathcal{T}$, $W_t = (R\tilde{a})^t W_0$ and (using the assumption that $\delta R^{1-\rho} < 1$)

$$\begin{aligned} U\left((a_s)_{s=t}^{\infty}\right) &= (1-\tilde{a})^{1-\rho}\left(1+\beta\sum_{s=t+1}^{\infty}\left(\delta\left(R\tilde{a}\right)^{1-\rho}\right)^{s-t}\right) \\ &= (1-\tilde{a})^{1-\rho}\left(1+\beta\left(-1+\sum_{s=0}^{\infty}\left(\delta\left(R\bar{a}\right)^{1-\rho}\right)^{s}\right)\right) \\ &= (1-\tilde{a})^{1-\rho}\left(1+\frac{\beta\delta\left(R\tilde{a}\right)^{1-\rho}}{1-\delta\left(R\tilde{a}\right)^{1-\rho}}\right) \\ &\equiv V\left(\tilde{a}\right). \end{aligned}$$

Next, suppose a constant savings rate of $\tilde{a}$ IP-dominates a constant savings rate of $\hat{a}$. Because it is at least as good as of $t = 0$ when wealth is the same, this implies that $V(\tilde{a}) \geq V(\hat{a})$, which immediately yields inequality 2 where wealth is also identical in the comparison. Hence proposition 2 establishes that these other paths are StPo-solutions as well, although as such, they must be supported by more "complex" strategies involving history-dependence (see Laibson [1994, Chapter 1]).

Indeed, I conjecture that, more generally, example 2 satisfies the following "regularity" property.

Definition 6. A decision problem satisfying history-independence is *welfare-regular* if, whenever a path $\omega = (a_t)_{t=0}^{T-1}$ is not IP-optimal, there exists a path $\omega' = (a'_t)_{t=0}^{T-1}$ which IP-dominates ω and, moreover, is such that, for any time $t \in \mathcal{T}$,

$$U_t\left(\omega'\right) \geq U_t\left(\left(a'_s\right)_{s=0}^{t-1}, \left(a_s\right)_{s=t}^{T-1}\right). \tag{3}$$

Welfare regularity restricts the history-dependence of welfare: if a path ω is not IP-optimal, then there is some other path ω' that IP-dominates it, where as long as ω' has been followed, the DM would never prefer switching to continuation as under ω over staying on ω'. Observe the similarity of inequalities (2) and (3), and note that the example given at the outset of this discussion violates welfare-regularity. Of course, welfare-regularity is weaker than history-independence in a welfare sense.

Remark 3. If a decision problem satisfies history-independence even in a welfare sense, then it is welfare-regular.

Proof. Simply note that when history-independence is satisfied in a welfare sense, in the above definition, $U_t\left((a'_s)_{s=0}^{t-1}, (a_s)_{s=t}^{T-1}\right) = U_t(\omega)$, whence IP-dominance immediately yields the inequality. □

Corollary 1. *Assume the decision problem satisfies history independence and is welfare-regular. Then, if a StPo-solution is not IP-optimal, it is IP-dominated by another StPo-solution.*

Proof. Let $\hat{\omega} = (\hat{a}_t)_{t=0}^{T-1}$ be a StPo-solution, where a path $\tilde{\omega} = (\tilde{a}_t)_{t=0}^{T-1}$ IP-dominates $\hat{\omega}$. Because the decision problem is welfare-regular, it is without loss of generality to choose $\tilde{\omega} = (\tilde{a}_t)_{t=0}^{T-1}$ such that inequality (2) holds true, whence it is a StPo-solution. □

Based on this corollary, I conjecture that every non-IP-optimal StPo-solution in example 2 is in fact IP-dominated by another StPo-solution (so that this is not only true about constant-rate paths).

In any case, this result immediately implies that if a decision problem satisfying history-independence which is welfare-regular has a unique StPo-solution, then this solution is IP-optimal. Moreover, under standard "well-behavedness" assumptions (e.g. compact action spaces and continuous utility functions), where IP-dominance of a path comes with the existence of an IP-optimal path that IP-dominates it, there then exists an IP-optimal StPo-solution.

4 Conclusion

This note addresses two important welfare phenomena in decision problems with time-inconsistent preferences: Pareto-inefficiency of StPo-solutions and IP-rankable multiplicity of such solutions. In a framework that allows for history-dependent welfare, my first result delineates the forms of intertemporal conflict inherent in preferences that yield inefficient outcomes in the Pareto-sense by showing that they must violate essential consistency whenever the horizon is finite and there is no indifference. Essential consistency is in fact necessary in a simple version of the "timing problem" analysed by O'Donoghue and Rabin [1999] where rewards are immediate and costs are delayed. While the discussion points out the likely obstacles to generalisations of these results even within the framework that this note assumes, because truncation is a popular approach to selection among multiple StPoE (see e.g. Laibson [1997]), finite-horizon results about welfare are also of interest for work on infinite-horizon problems.

The property of essential consistency was proposed by Hammond [1976] for a similar decision environment, where he discovered it to be sufficient for the coincidence of naïve and sophisticated choice. An interesting question is therefore the more general relationship between Pareto-efficiency and this invariance property of choice to various degrees of preference misprediction.

On the other hand, when some StPo-solution in a decision problem satisfying history-dependence fails to be IP-optimal, then this comes with IP-rankable multiplicity *of StPo-solutions* when the effects of past play on welfare satisfy a certain regularity property. The latter kind of multiplicity appears to have played a major role for the development of refinements of StPoE, but whereas the work in this area so far has relied mostly on rather abstract and specific examples (of the class described) to promote their own respective approaches, I thus organise them into a general insight. Moreover, beyond such abstract examples, my result applies also to the influential consumption-savings model of Phelps and Pollak [1968].

Maybe most importantly, however, the last result can be used to establish existence of IP-optimal StPo-solutions under standard technical assumptions. To the best of my knowledge, no such result has been available. Of course, its generalisation to a broader class of problems would be highly desirable for applications.

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