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Abstract

Second-best incentive compatible allocation rules for multiple-type indivisible objects

by Hidekazu Anno and Morimitsu Kurino*

We consider the problem of allocating several types of indivisible goods when preferences are separable and monetary transfers are not allowed. Our finding is that the coordinate-wise application of strategy-proof and non-wasteful rules yields a strategy-proof rule with the following efficiency property: no strategy-proof rule Pareto-dominates the rule. Such rules are abundant as they include the coordinate-wise use of the two well-known priority-based rules of the top trading cycles (TTC) and the deferred acceptance (DA). Moreover, our result supports the current practice in Market Design that separately treats each type of market for its design.

Keywords: Strategy-proofness, second-best incentive compatibility, top trading cycles rules, deferred acceptance rules

JEL classification: C78, D47, D71

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1 Introduction

We consider an indivisible goods resource allocation problem without monetary transfers. The simplest and basic model in the literature is a house allocation problem (Shapley and Scarf, 1974; Hylland and Zeckhauser, 1979) where one type of object group (indivisible goods) is to be allocated and each agent consumes exactly one object. Its notable real-life applications have been proposed for on-campus housing assignments for college students (Abdulkadiroğlu and Sönmez, 1999), school choice problems (Abdulkadiroğlu and Sönmez, 2003), and for kidney exchange for patients (Roth, Sönmez, and Ünver, 2004).

However, in many real-life resource allocation problems, we often deal with multiple types of objects at the same time. For example, many families with children live in public housing, and their children go to school. Here one type of object is public housing, while another is seats in schools. The percentage of the population living in public housing is about 10 to 35 in many countries.[1] This suggests that about the same percentage of families with children would participate in two types of markets - public housing assignment and school choice programs.[2] The public housing assignment can be modeled as house allocation with existing tenants (Abdulkadiroğlu and Sönmez, 1999) where in addition to new applicants for housing, existing tenants occupy houses and can swap housing.[3] Most districts in various countries use the serial dictatorship rule for public housing which is *strategy-proof* and *Pareto efficient*. Under the rule the priority is given by the degree of need, income levels, or is drawn by a lottery, and then the families with the highest priority are assigned apartments. On the other hand, since Abdulkadiroğlu and Sönmez (2003) adovocated the use of *strategy-proof* and *efficient* (or at least *non-wasteful*) rules,[4] more and more districts in various countries have adopted school choice programs where parents can "choose" public schools for their children. Therefore a potentially large number of families participate in public housing assignments and school choice programs.

One important observation is that in most cases, if not all, assignment procedures for different types of objects are operated by different government authorities, and each procedure is independent from the other.[5] Moreover, the market design literature treats each market independently, and then considers the design of *stratetgy-proof* and *efficient* (or at least *non-wasteful*) rules for only one type of market. However, such independent operation of different marekts leads to inefficiency when we

[1] For example, Whitehead and Scanlon (2007) report that the percentages in European countries around 2000 are 35 in the Netherlands, 25 in Austria, 21 in Denmark, 20 in Sweden, 18 in England, 17 in France, 8 in Ireland, 6 in Germany, and 4 in Hungary. Moreover, the percentages are 30 in Hong Kong and 9.9 in Japan (The webpages are http://www.housingauthority.gov.hk/en/public-housing/index.html and http://www.stat.go.jp/data/jyutaku/2008/10_3.htm. The webs were accessed on December 18, 2013.)

[2] The precise data is not available, but there is an exception. Schwartz, McCabe, Ellen, and Chellman (2010) report that 14 % of students in public elementary and middle schools were living in public housing in New York City in 2002.

[3] For example, in the UK, tenants have the opportunity to swap houses (Department for Communities and Local Government, 2013).

[4] Under a *non-wasteful* allocation (Balinski and Sönmez, 1999), if some object is preferred to the assigned one for an agent, it is fully assigned up to its quota. Thus, *non-wastefulness* is an efficiency axiom weaker than *Pareto efficiency*.

[5] For example, in the City of Boston, the school choice assignment is operated by Boston Public Schools, while the public housing assignment is by the Boston Housing Authority.

take all types of markets into account. To the best of our knowledge, this paper is the first to provide theoretical support for such practices of treating each type market independently and using a *strategy-proof* and *non-wasteful* rule for each type.

In this paper, we focus on a model with multiple types of objects (multiple markets) when each agent has a separable preference, where a preference over bundles is *separable if* there is a list of preferences over each single-type market such that if two bundles x and y are different in only t-th type, then the evaluation between x and y coincides with the evaluation of t-th types x^t and y^t according to the preference over the t-th market objects.[6]

One of the most serious difficulties pertaining to the multiple types arises from the tension between incentive compatibility and efficiency. It is known that the class of *strategy-proof* and *Pareto efficient* rules in a multiple-type model is extremely narrower than the one in a single-type model. In particular, the requirement of the two properties results in a serial dictatorship rule in which each agent chooses her assignment one by one according to an exogenously fixed priority order (Monte and Tumennasan, 2013).[7] However, such a rule is against the independent operation of markets in real life and is extremely unfair.[8]

With this difficulty, we take a natural research direction of looking for a plausible rule by relaxing *Pareto efficiency* while keeping *strategy-proofness*. That is, we search for "*second-best (efficient) incentive compatible*" rules. To be precise, we call a rule *second-best incentive compatible if* the rule is itself *strategy-proof*, and is not Pareto-dominated by any other *strategy-proof* rule.[9] It is a quite natural second-best efficiency concept as far as we are concerned with *strategy-proofness* since the notion is a straightforward adaptation of the Pareto criterion for the class of *strategy-proof* rules. This paper provides a simple sufficient condition for the *second-best incentive compatibility*.

To state our main result, it is worth noting that there is a very simple way to construct a *strategy-*

[6] A model with single-type objects is a special case of ours. Although our main focus is a multiple-type model, all of our results remain true in single-type models. There are a few papers on multiple-type indivisible goods resource allocation problems: Konishi, Quint, and Wako (2001) and Klaus (2008) study a problem with endowments, referred to as multiple-type housing markets, while similar to our paper Monte and Tumennasan (2013) examine a problem without endowments.

[7] Rigorously speaking, Monte and Tumennasan's (2013) model is not a special case of ours due to the difference in preference domains (See Assumption 2 and footnote 16). They prove that *if* a rule is *strategy-proof*, *Pareto efficient* and *non-bossy*, then it is a sequential dictatorship rule that is a variant of the serial dictatorship rule.

The same kind of difficulty is observed in many models including pure exchange economies (Serizawa, 2002), public goods economies (Zhou, 1991), two-sided matching problems (Alcalde and Barbera, 1994), and dynamic matching problems (Kurino, 2014).

[8] Consider the environment with homogeneous preferences of agents and unit quotas of objects. In the "full" serial dictatorship (SD) rule, the highest-priority agent receives her favorite objects from all types, and is envied by all agents in each type. However, we can mitigate the unfairness with a "market-wise" serial dictatorship (SD) rule where a priority is defined for each type, and the highest-priority agent in a type is different from those in the other markets. In this case for each agent there is only one type market for which she is envied by all agents. Although it is Pareto inefficient (Section 3.4), the market-wise SD rule to a large extent remedies the unfair feature of the full SD rule.

[9] To the best of our knowledge, a variant of *second-best incentive compatibility* was first studied in Sasaki (2003) in the context of divisible resource allocation problems with multi-dimensional single-peaked preferences. See also Anno and Sasaki (2013) for the same model. Moreover, Klaus (2008) investigates the *second-best incentive compatibility* for multiple-type indivisible goods resource allocation problems, while Abdulkadiroğlu, Pathak, and Roth (2009); Erdil (2011); Kesten and Kurino (2013) study for single-type problems.

proof rule in our setting with multiple-type objects. Namely, we first consider an independent rule which applies a single-type rule in each market, and then consider only *strategy-proof* single-type rules for an independent rule. This is made possible due to the separability of preferences. Based on this, our main result (Theorem 1) is the following: *An independent rule obtained by the market-wise application of strategy-proof and non-wasteful single-type rules is second-best incentive compatible.*

Now, let us explain the economic implications of Theorem 1. Inspired by Gale's top trading cycles (Shapley and Scarf, 1974) and Gale and Shapley's (1962) deferred acceptance algorithm, the market design literature has uncovered *strategy-proof* and *non-wasteful* rules for various single-type matching problems such as school choice problems, on-campus housing assignment, kidney exchange problems for patients (See Section 5). However, even if we successfully design such a *strategy-proof* and *Pareto efficient* rule for each single-type market, the resulting rules may collectively fail to be *Pareto efficient* (See Section 3.4). Theorem 1 guarantees that these practices of designing such a rule lead to *second-best incentive compatiblility* - the impossibility of being Pareto-improved without sacrificing *strategy-proofness.* Moreover, we have a rich class of *second-best incentive compatible* rules, since most single-type rules, if not all, proposed in the literature are *strategy-proof* and *non-wasteful.* The positive result and the richness of rules are in marked contrast to requiring the "*first-best incentive compatibility*" (namely, the combination of *strategy-proofness* and *Pareto efficiency*) which often results in an extremely unfair rule of serial dictatorship in our multiple-type setting. That is, the efficiency loss in these rules is an inevitable cost of recovering the fairness and the independent operation of markets as long as we preserve *strategy-proofness.*

It is important to note that an analogous difficulty is observed in the dynamic matching literature, too.[10] Kurino (2014) and Kennes, Monte, and Tumennasan (2013) observe that we often have *dynamic Pareto inefficiency* when we repeat statically *strategy-proof* and *Pareto efficient* rules in each period. Moreover, they notice that dynamic efficiency can be achieved by an extremely unfair rule of letting the highest-priority agent have her best objects in *all* periods, the second-highest-priority agent have her best among those remaining, and so on. Since one of the interpretations of our general setup is a dynamic matching problem, our result is also useful in this context.[11]

This paper consists of six sections. In section 2 we present the model and the axioms we are interested in. In section 3 we define top trading cycles rules and deferred acceptance rules. In section 4 we present the main results. In section 5 briefly discuss market design applications. Section 6 concludes. All proofs are relegated to the Appendix.

[10]Kurino (2014) incorporates the overlapping generations structure into a house allocation problem with an on-campus housing assignment as the key applications. Kennes, Monte, and Tumennasan (2013) and Pereyra (2013) extend his model to study a dynamic school choice problem.

[11]In this interpretation, the set of types is viewed as the set of periods. Since we assume that the set of types is finite, our model can accommodate a dynamic matching model with a finite horizon.

Table 1: An example

	Type 1	Type 2	Type 3	
Agent 1	$\surd$			$T_1 = \{1\}$
Agent 2	$\surd$	$\surd$		$T_2 = \{1, 2\}$
Agent 3		$\surd$	$\surd$	$T_3 = \{2, 3\}$
	$N^1 = \{1, 2\}$	$N^2 = \{2, 3\}$	$N^3 = \{3\}$	

Note: In this example, there are three agents and three types. The symbol $\surd$ indicates which agent is interested in what types of objects. For example, agent 2 is interested in both types 1 and 2.

2 Model and Axioms

2.1 Multiple-type markets

We introduce a general model of multiple-type markets where an agent is interested in being assigned multiple types of objects when monetary transfers are not allowed. A **multiple-type market** is a list $(N, T, (T_i)_{i\in N}, (X^t)_{t\in T}, q, (R_i)_{i\in N})$: $N := \{1, \ldots, n\}$ is a finite set of agents with $|N| \geq 2$, while T is a finite set of types of objects (i.e., indivisible goods). We identify T with $\{1, \cdots, |T|\}$. If $|T| = 1$, we call a market **single type**. For each type $t \in T$, objects of type-t are available. Let X^t be a finite set of type-t objects with $|X^t| \geq 2$. For each type $t \in T$, $q(x^t)$ is the **quota** of type-t object $x^t \in X^t$. That is, q is a function from $\cup_{t\in T} X^t$ to $\mathbb{Z}_{++}$ and $q(x^t)$ indicates the number of identical type-t object x^t. Each agent $i \in N$ is interested in at least one type of objects, and consumes one object from each type in which she is interested. Let $T_i \in 2^T \backslash \{\emptyset\}$ be the set of types in which agent i is interested (See Table 1). Moreover, for each type $t \in T$, let $N^t := \{i \in N | t \in T_i\}$ be the set of agents who are interested in type-t objects. We assume that for each $t \in T$, at least one agent is interested in the type-t, i.e., $N^t \neq \emptyset$. Throughout the paper we maintain the following assumption.[12]

Assumption 1. (Existence of null objects) *For each type $t \in T$, there is the null object, denoted by $\emptyset^t$, in X^t which satisfies $q(\emptyset^t) \geq |N^t|$.*[13]

The null object represents an outside option. Since each agent consumes one object from each of her interested types, the condition ($q(\emptyset^t) \geq |N^t|$) implies that the null object $\emptyset^t$ is sufficiently available so that it can be consumed by all agents interested in the type-t market at the same time. We discuss in the conclusion how this assumption affects our main result.

Finally, we describe the preferences of agents. Since each agent $i \in N$ consumes one object from each type in T_i, her consumption space is $X_i := \prod_{t\in T_i} X^t$. An element of X_i is called a **bundle**, generically denoted by $x_i = (x_i^t)_{t\in T_i} \in X_i$. For convenience and clarity, we introduce some notations: For a finite set Y, let $\mathcal{R}(Y)$ be the set of all complete and transitive binary relations on Y, and $\mathcal{P}(Y)$

[12]The only exception is the paragraph right before Corollary 3. There we do not assume Assumption 1.
[13]An object is called real *if* it is not null.

the set of all complete, transitive, and anti-symmetric binary relations on Y.[14] Each agent $i \in N$ is equipped with a preference relation R_i on X_i, i.e., $R_i \in \mathcal{R}(X_i)$. For each $R_i \in \mathcal{R}(X_i)$, P_i and I_i denote the asymmetric and symmetric parts of R_i. We denote $R := (R_i)_{i \in N} \in \prod_{i \in N} \mathcal{R}(X_i)$ and call it a **profile**. For each profile $R = (R_1, \cdots, R_n) \in \prod_{i \in N} \mathcal{R}(X_i)$, and each $i \in N$, the subprofile obtained by removing i's preference is denoted by R_{-i}; that is, $R_{-i} := (R_1, \cdots, R_{i-1}, R_{i+1}, \cdots, R_n)$. It is convenient to write the profile $(R_1, \cdots, R_{i-1}, R'_i, R_{i+1}, \cdots, R_n)$ as $(R'_i; R_{-i})$.

Now, we introduce three classes of preferences. First, a preference $R_i \in \mathcal{R}(X_i)$ is **separable** *if* for each $t \in T_i$ there is a preference $R_i^t \in \mathcal{P}(X^t)$ on type-t objects such that for each pair of bundles $\{x_i, x'_i\} \subseteq X_i$,

$$[\text{ for each } t \in T_i, x_i^t \ R_i^t \ x_i'^t] \Rightarrow x_i \ R_i \ x'_i.$$

We denote by $\mathcal{R}_{sep}(X_i)$ the set of all separable preferences on X_i, and call R_i^t a **type-t preference**. Let $\mathcal{P}_{sep}(X_i)$ be the set of all separable strict preferences on X_i, i.e., $\mathcal{P}_{sep}(X_i) = \mathcal{R}_{sep}(X_i) \cap \mathcal{P}(X_i)$.

To introduce the second class of preferences, for each $t \in T$, let $\mathcal{U}^t$ be the set of strict utility functions on X^t. That is, $u_i^t : X^t \to \mathbb{R}$ belongs to $\mathcal{U}^t$ *if* u_i^t is injective. A preference $R_i \in \mathcal{R}(X_i)$ is **additively separable** *if* for each $t \in T_i$ there is a utility function $u_i^t \in \mathcal{U}^t$ on type-t objects such that for each pair of bundles $\{x_i, x'_i\} \subseteq X_i$,

$$x_i \ R_i \ x'_i \Leftrightarrow \sum_{t \in T_i} u_i^t(x_i^t) \geq \sum_{t \in T_i} u_i^t(x_i'^t).$$

We denote by $\mathcal{R}_{add}(X_i)$ the set of all additively separable preferences on X_i, and by $\mathcal{P}_{add}(X_i)$ the set of all additively separable strict preferences on X_i, i.e., $\mathcal{P}_{add}(X_i) = \mathcal{R}_{add}(X_i) \cap \mathcal{P}(X_i)$.

To describe the third class, let $\Sigma(T_i)$ be the set of bijections from $\{1, \cdots, |T_i|\}$ to T_i. A preference $R_i \in \mathcal{R}(X_i)$ is **lexicographic** *if* there are $\sigma \in \Sigma(T_i)$ and a list of type preferences $(R_i^t)_{t \in T_i} \in \prod_{t \in T_i} \mathcal{P}(X^t)$ such that for each pair of bundles $\{x_i, x'_i\} \subseteq X_i$ with $x_i \neq x'_i$,

$$x_i \ R_i \ x'_i \Leftrightarrow x_i^{\sigma(1)} \ P_i^{\sigma(1)} \ x_i'^{\sigma(1)} \text{ or } \Big[\exists t \in \{1, \cdots, |T_i|\} \backslash \{1\} \text{ s.t.} \{\forall t' < t, x_i^{\sigma(t')} = x_i'^{\sigma(t')}\} \text{ and } x_i^{\sigma(t)} \ P_i^{\sigma(t)} \ x_i'^{\sigma(t)}\Big].$$

We denote by $\mathcal{P}_{lex}(X_i)$ the set of all lexicographic preferences on X_i. The next remark immediately follows from the definitions.

Remark 1. Let $i \in N$.

1. For each $R_i \in \mathcal{P}_{sep}(X_i) \cup \mathcal{R}_{add}(X_i)$, the list of corresponding type-t preferences $(R_i^t)_{t \in T_i}$ is unique.

2. We have the following relations among the classes of preferences.[15]

[14] A binary relation $\geq$ is complete *if* for each $\{y, y'\} \subseteq Y$, $y \geq y'$ or $y' \geq y$. A binary relation $\geq$ is transitive *if* for each $\{y, y', y''\} \subseteq Y$, $y \geq y'$ and $y' \geq y''$ imply $y \geq y''$. A binary relation $\geq$ is anti-symmetric *if* for each $\{y, y'\} \subseteq Y$, $y \geq y'$ and $y' \geq y$ imply $y = y'$.

[15] Non-trivial statements are $\mathcal{P}_{lex}(X_i) \subseteq \mathcal{P}_{add}(X_i)$, $\mathcal{P}_{add}(X_i) \neq \mathcal{P}_{sep}(X_i)$, and $\mathcal{R}_{add}(X_i) \neq \mathcal{R}_{sep}(X_i)$. The proof of the third can be found on page 43 in Fishburn (1970). The other two are proved in Appendix A.

$$\begin{array}{ccccc} \mathcal{P}_{lex}(X_i) & \subsetneq & \mathcal{P}_{add}(X_i) & \subsetneq & \mathcal{P}_{sep}(X_i) \\ & & \cap\!\!\!\!\not\;\; & & \cap\!\!\!\!\not\;\; \\ & & \mathcal{R}_{add}(X_i) & \subsetneq & \mathcal{R}_{sep}(X_i) \end{array}$$

Let $\mathcal{D}_i$ be the set of agent i's admissible preferences. Let $\mathcal{D} := \prod_{i \in N} \mathcal{D}_i$ be the set of admissible profiles. In the rest of the paper we keep the following assumption.

Assumption 2. (Admissible preferences) *For each $i \in N$, $\mathcal{P}_{add}(X_i) \subseteq \mathcal{D}_i \subseteq \mathcal{P}_{sep}(X_i) \cup \mathcal{R}_{add}(X_i)$.*

Our domain covers two types of wide ranges of domains: When one type consists of separable strict preferences, it ranges from the additively separable domain to the universal one (See Item 2 in Remark 1, in particular $\mathcal{P}_{add}(X_i) \subseteq \mathcal{P}_{sep}(X_i)$). The other type ranges from the strict domain to the weak one when it consists of additively separable preferences (See Item 2 in Remark 1, in particular $\mathcal{P}_{add}(X_i) \subseteq \mathcal{R}_{add}(X_i)$). As long as we keep a separability of preferences, ours is the most natural and covers the widest range of domains in the literature.[16] This completes the description of a multiple-type market $(N, T, (T_i)_{i \in N}, (X^t)_{t \in T}, q, (R_i)_{i \in N})$. We assume throughout the paper that N, T, $(T_i)_{i \in N}$, $(X^t)_{t \in T}$, and q are fixed.

Let us comment on the model. Our model allows for agents' partial participation on type markets ($T_i \subseteq T$) and covers the full participation on all type markets (for each $i \in N$, $T_i = T$) considered in the literature.[17]

2.2 Type markets

Let a multiple-type market $(N, T, (T_i)_{i \in N}, (X^t)_{t \in T}, q, (R_i)_{i \in N})$ be given. Then, by Assumption 2 and Item 1 in Remark 1, for each $i \in N$ and each $R_i \in \mathcal{D}_i$, each of the corresponding type-t preference is unique and strict, and thus denoted by R_i^t. Thus, it makes sense to introduce the type-t market induced from the multiple-type market. The induced **type-t market** is the single-type market $(N^t, X^t, q|_{X^t}, (R_i^t)_{i \in N^t})$ where each agent $i \in N^t$ consumes one type-t object in X^t; $q|_{X^t}$ is the restriction of q to X^t which indicates the quota $q(x^t)$ of each type-t object $x^t \in X^t$. Finally, R_i^t is the type-t preference of agent i in $\mathcal{P}(X^t)$. We denote $R^t := (R_i^t)_{i \in N^t} \in \mathcal{P}(X^t)^{N^t}$ and call it the **type-t profile** induced from $(R_i)_{i \in N}$, or just the **type profile** induced from $(R_i)_{i \in N}$.[18] Similarly to the notations of profiles, we use $R_{-i}^t = (R_1^t, \cdots, R_{i-1}^t, R_{i+1}^t, \cdots, R_n^t)$ and $(R_i'^t; R_{-i}^t) = (R_1^t, \cdots, R_{i-1}^t, R_i'^t, R_{i+1}^t, \cdots, R_n^t)$.

Note that our single-type market or type market, $(N^t, X^t, q|_{X^t}, (R_i^t)_{i \in N^t})$, is the traditional house allocation problem (Hylland and Zeckhauser, 1979), which can include the following matching problems. See Sönmez and Ünver (2011) for a comprehensive survey on the subject.

[16]See Konishi, Quint, and Wako (2001); Klaus (2008); Monte and Tumennasan (2013) whose domain is the set of separable strict preferences, $\mathcal{P}_{sep}(X_i)$. An exception is Kurino (2014) who allows weak preferences, though his dynamic model is slightly different from the multiple-type goods model. Strictly speaking, Monte and Tumennasan's (2013) domain additionally assumes that the null bundle is worst.

[17]See Konishi, Quint, and Wako (2001); Klaus (2008); Monte and Tumennasan (2013).

[18]In the list $(N^t, X^t, q|_{X^t}, (R_i^t)_{i \in N^t})$, for simplicity we omit the type set and the structure of interested types. Note that in the type-t market, the type set is $\{t\}$ and every agent $i \in N^t$ is interested in type-t.

- *House allocation with existing tenants.* Abdulkadiroğlu and Sönmez (1999) introduce the problem that deals with the on-campus housing assignments for U.S. college students. The problem can be applied not only to the on-campus housing assignment but also the public housing assignment. In this setting, the set N^t of agents is divided into the set N^t_N of newcomers and the set N^t_E of existing tenants, while objects refer to houses. Corresponding to each existing tenant $i \in N^t_E$ is a unique occupied house $\omega^t_i \in X^t \setminus \{\emptyset^t\}$ which is interpreted as the house that agent i currently lives in. Each newcomer is assumed to occupy the null object $\emptyset^t$.

- *Kidney exchange.* Transplantation is the preferred treatment for patients diagnosed with end-stage kidney disease. Roth, Sönmez, and Ünver (2004) introduce the problem that aims to efficiently organize direct exchanges among medically incompatible donor-patient pairs as well as indirect exchanges that involve one incompatible donor-patient and deceased donors. In the setting, agents refer to patients and objects refer to kidneys. Each patient $i \in N^t$ is paired with a kidney $k_i \in X^t \setminus \{\emptyset^t\}$ with $q(k_i) = 1$ supplied by her intended donor. Thus $|N^t| = |X^t \setminus \{\emptyset^t\}|$. Every patient also has the option to enter the waiting list for cadaveric kideneys with priority. Let the null object refer to this option and denote it by w.[19]

- *School choice.* Abdulkadiroğlu and Sönmez (2003) introduce the problem that concerns student assignment to public schools. In this context agents refer to the students, objects refer to the schools, and the null object represents other education options such as private schools. Additionally, each school $x^t \in X^t$ has a priority $\succeq^t_{x^t}$ over students which is a linear order in $\mathcal{P}(N^t)$.

2.3 Rules

A **type-t allocation** is a function a^t from N^t to X^t such that each agent $i \in N^t$ is assigned type-t object a^t_i, and for each type-t object $x^t \in X^t$ the number of agents who are assigned x^t does not exceed the quota $q(x^t)$, i.e., $|\{i \in N^t | a^t_i = x^t\}| \leq q(x^t)$. Let $\mathcal{A}^t$ be the set of all type-t allocations. An **allocation**, consisting of type allocations, is $a := (a^1, \cdots, a^{|T|}) \in \prod_{t \in T} \mathcal{A}^t$ where for each $t \in T$, a^t is a type-t allocation. Let $\mathcal{A} := \prod_{t \in T} \mathcal{A}^t$ be the set of all allocations. Given $a \in \mathcal{A}$, for each $i \in N$, let a_i be the agent i's bundle at a, i.e., $a_i := (a^t_i)_{t \in T_i}$.

We focus on a deterministic rule in this paper. A **rule** selects an allocation for each profile in a multiple-type market, i.e., it is a function $\varphi : \mathcal{D} \to \mathcal{A}$. For each $R \in \mathcal{D}$, $\varphi_i(R)$ denotes the agent i's bundle at $\varphi(R)$, and $\varphi^t(R)$ denotes the type-t allocation at $\varphi(R)$. On the other hand, a **type-t rule** selects a type-t allocation for each type-t profile, i.e., it is a function $\Phi^t : \mathcal{P}(X^t)^{N^t} \to \mathcal{A}^t$. For each $R^t \in \mathcal{P}(X^t)^{N^t}$, $\Phi^t_i(R^t)$ denotes the agent i's type-t object at $\Phi^t(R^t)$. Note that $\Phi^t(R^t)$ depends only on the preferences of N^t while $\varphi^t(R)$ may depend on the preferences of $N \setminus N^t$. In the conclusion we

[19] Unlike the original framework of Roth, Sönmez, and Ünver (2004) that allows for heterogeneous preferences, later models of the kidney exchange deal with dichotomous preferences (e.g., Roth, Sönmez, and Ünver, 2005; Yilmaz, 2011).

discuss how our main result can be affected for a lottery rule.

Given a list of type rules, $(\Phi^t)_{t\in T}$, we can naturally define a rule for bundles due to separable preferences:

Definition 1. A rule φ is **independent** *if* there exists a list of type rules $(\Phi^t)_{t\in T}$ such that for each $R \in \mathcal{D}$ and each $t \in T$, $\varphi^t(R) = \Phi^t(R^t)$. If such a $(\Phi^t)_{t\in T}$ exists, it is unique. Thus, if a rule φ is independent, we denote its corresponding type-t rule by Φ^t.

An independent rule, φ, treats each type market independently and separately in that a type-t allocation under φ depends only on type-t preference profiles. Note also that an independent rule is less informationally demanding than a dependent one, because the former only requires type preferences and the latter requires preferences on bundles that contain type preferences.

Finally, we introduce the dominations of rules: An allocation $a \in \mathcal{A}$ **(Pareto) dominates** an allocation $b \in \mathcal{A}$ at $R \in \mathcal{D}$, written as a dom(R) b, *if* for each $i \in N$, $a_i \; R_i \; b_i$, and for some $i \in N$, $a_i \; P_i \; b_i$. Similarly, a type-t allocation $a^t \in \mathcal{A}^t$ **dominates** a type-t allocation $b^t \in \mathcal{A}^t$ at $R^t \in \mathcal{P}(X^t)^{N^t}$, written as a^t dom(R^t) b^t, *if* for each $i \in N^t$, $a^t_i \; R^t_i \; b^t_i$, and for some $i \in N^t$, $a^t_i \; P^t_i \; b^t_i$. Now we can define the domination of rules: A rule φ **dominates** another rule ζ, written as φ dom ζ *if* for each $R \in \mathcal{D}$ and each $i \in N$, $\varphi_i(R) \; R_i \; \zeta_i(R)$, and for some $R \in \mathcal{D}$, $\varphi(R)$ dominates $\zeta(R)$ at R.

2.4 Axioms

We introduce axioms for both rules and type rules.

The first is an incentive compatibility axiom that says no agent can benefit from misreporting her preference. A rule φ is **strategy-proof** *if* for each $R \in \mathcal{D}$, each $i \in N$ and each $R'_i \in \mathcal{D}_i$, $\varphi_i(R) \; R_i \; \varphi_i(R'_i; R_{-i})$. Similarly, a type-$t$ rule Φ^t is **strategy-proof** *if* for each $R^t \in \mathcal{P}(X^t)^{N^t}$, each $i \in N^t$ and each $R'^t_i \in \mathcal{P}(X^t)$, $\Phi^t_i(R^t) \; R^t_i \; \Phi^t_i(R'^t_i; R^t_{-i})$.

The second is an efficiency axiom that says, for each profile, the selected allocation should not be dominated by any other allocation at the profile. A rule φ is **Pareto efficient** *if* for each profile $R \in \mathcal{D}$, there is no allocation $a \in \mathcal{A}$ such that a dominates $\varphi(R)$ at R. Similarly, a type-t rule Φ^t is **Pareto efficient** *if* for each type-t profile $R^t \in \mathcal{P}(X^t)^{N^t}$, there is no type-$t$ allocation $a^t \in \mathcal{A}^t$ such that a^t dominates $\Phi^t(R^t)$ at R^t.

The third is a weak efficiency axiom defined only for a type rule. A type-t rule Φ^t is **non-wasteful** (Balinski and Sönmez, 1999) *if* for each $R^t \in \mathcal{P}(X^t)^{N^t}$, each $i \in N^t$ and each $x^t \in X^t$, $x^t \; P^t_i \; \Phi^t_i(R^t)$ implies $|\{j \in N^t | \Phi^t_j(R^t) = x^t\}| = q(x^t)$.

The last one is a very weak form of a voluntary participation axiom that says for each profile no agent can be worse off than nothing. A rule φ is **individually rational** *if* for each $R \in \mathcal{D}$ and each $i \in N$, $\varphi_i(R) \; R_i \; (\emptyset^t)_{t\in T_i}$. Moreover, a type-$t$ rule Φ^t is **individually rational** *if* for each $R^t \in \mathcal{P}(X^t)^{N^t}$ and each $i \in N^t$, $\Phi^t_i(R^t) \; R^t_i \; \emptyset^t$.

The following results are straightforward from definitions.

Remark 2. Let Φ^t be a type-t rule.

1. If Φ^t is *Pareto efficient*, then Φ^t is *non-wasteful*.

2. If Φ^t is *non-wasteful*, then Φ^t is *individually rational*.

Remark 3. Suppose that a rule φ is independent.

1. If for each $t \in T$, Φ^t is *strategy-proof*, then φ is *strategy-proof*.

2. If for each $t \in T$, Φ^t is *individually rational*, then φ is *individually rational*.

Note that the converse may not be always true in each statement in Remarks 2 and 3 except for Item 1 in Remark 3.

3 Three Classes of Priority-based Rules

We first introduce a priority profile, and then the three priority-based rules that have played central roles in the literature. The single-type market with priority profiles is called a school choice problem or a priority-based indivisible goods resource allocation problem in the literature.

A priority is defined for each type-t object that orders all agents who are interested in the type-t market and expresses how each agent is treated for the object. Formally, a **priority** of type-t object $x^t \in X^t$ is a linear order in $\mathcal{P}(N^t)$, denoted by $\succeq^t_{x^t}$. We denote $\succeq^t := (\succeq^t_{x^t})_{x^t \in X^t} \in \mathcal{P}(N^t)^{X^t}$ and call it a **type-t priority profile**. Moreover, we denote $\succeq := (\succeq^t)_{t \in T} \in \prod_{t \in T} \mathcal{P}(N^t)^{X^t}$ a **priority profile**.

3.1 Market-wise top trading cycles (TTC) rule

The top trading cycles (TTC) rule for a single-type market is introduced by Abdulkadiroğlu and Sönmez (2003) who modify Gale's top trading cycles described in Shapley and Scarf (1974). Given a type-t profile $R^t \in \mathcal{P}(X^t)^{N^t}$, the **top trading cycles (TTC) type-rule induced by a type-t priority profile $\succeq^t$**, denoted by $TTC^{\succeq^t}$, selects a type-t allocation as follows:

Step 1: Each agent points to the most favorite object according to her preference and each object points to the agent who has the highest priority for that object. Note that there is at least one cycle.[20] Each agent in a cycle receives the object she points to and is removed from the market. Each object in a cycle whose quota is one is also removed.

Step $k (\geq 2)$: Each agent who has not been removed in previous steps points to the most favorite object among the remaining objects according to her preference and each remaining object points to the agent who has the highest priority among the remaining agents for that object. Note that there

[20] A cycle is an ordered list of agents and objects $(i_1, x_1, i_2, x_2, \dots, i_m, x_m)$ such that i_1 points to x_1, x_1 points to i_2, i_2 points to x_2, ..., i_m points to x_m, and x_m points to i_1.

is at least one cycle. Each agent in a cycle receives the object she points to and is removed from the market. Each object in a cycle is also removed if the number of cycles containing the object formed through k steps is equal to the quota of the object.

The algorithm terminates when no agent remains in the market.

Remark 4. For each type $t \in T$ and each priority profile $\succeq^t \in \mathcal{P}(N^t)^{X^t}$, the TTC type-rule induced by $\succeq^t$ is *strategy-proof* and *Pareto efficient* (Abdulkadiroğlu and Sönmez, 2003).

Definition 2. The market-wise top trading cycles (TTC) rule induced by a priority profile $\succeq = (\succeq^t)$, denoted by $TTC^{\succeq}$, selects its type-t allocation as $TTC^{\succeq^t}(R^t)$ that is chosen by the TTC type rule induced by a type-t priority profile.

Remark 5. Note that by Remarks 2, 3, and 4, a market-wise TTC rule is *strategy-proof* and *individually rational*. However, we will show in subsection 3.4 that it is not *Pareto efficient* for some priority profiles and markets.

3.2 Market-wise deferred acceptance (DA) rule

The deferred acceptance (DA) rule for a single-type market is introduced by Abdulkadiroğlu and Sönmez (2003) who apply Gale and Shapley's (1962) agent-proposing deferred acceptance algorithm in a college admissions problem to an indivisible goods resource allocation problem. Given a type-t profile $R^t \in \mathcal{P}(X^t)^{N^t}$, **the deferred acceptance (DA) type-rule induced by a priority profile** $\succeq^t$, denoted by $DA^{\succeq^t}$, selects a type-t allocation as follows.

Step 1: Each agent applies to the most favorite object according to her preference. Each object selects agents from its applicants up to its quota according to its priority and tentatively keeps them. Any remaining agents are rejected.

Step $k(\geq 2)$: Each agent who was rejected in the previous step applies to her next favorite object according to her preference. Each object selects agents from its new applicants and the tentatively kept agents up to its quota according to its priority and tentatively keeps them. Any remaining agents are rejected.

The algorithm terminates when no agent is rejected.

Remark 6. For each type $t \in T$ and each type-priority profile $\succeq^t \in \mathcal{P}(N^t)^{X^t}$, the DA type-rule induced by $\succeq^t$ is *strategy-proof* and *non-wasteful* (Balinski and Sönmez, 1999).[21]

[21] Ergin (2002) characterizes the priority profiles under which the type-t DA rule is *Pareto efficient*. Kesten (2006) characterizes the priority profiles under which the type-t TTC rule is *fair* in terms of envies at the selected type-t allocation.

Definition 3. The market-wise deferred acceptance (DA) rule induced by a priority profile $\succeq = (\succeq^t)_{t \in T}$, denoted by $DA^{\succeq}$, selects its type-t allocation as $DA^{\succeq^t}(R^t)$ that is chosen by the DA type-rule induced by a type-t priority profile.

Remark 7. By Remarks 2, 3, and 6, a market-wise DA rule is *strategy-proof* and *individually rational.* However, we show in the subsection 3.4 that even if each DA type-rule is *Pareto efficient* at some priority profile, its market-wise DA is not *Pareto efficient.*

3.3 Market-wise serial dictatorship (SD) rule

The serial dictatorship (SD) rule for a type-t market with respect to a priority order $\geq^t \in \mathcal{P}(N^t)$, written as $SD^{\geq^t}$, is described as follows: for each type-t preference profile, letting the highest-priority agent with respect to $\geq^t$ have her best object, the second-highest-priority agent with respect to $\geq^t$ have her best among those remaining, and so on.

Note that given a priority order $\geq^t \in \mathcal{P}(N^t)$, the serial dictatorship rule for a type-t market with respect to $\geq^t$ coincides with the TTC and DA rules for a type-t market with the priority profile $\succeq^t \in \mathcal{P}(N^t)^{X^t}$ in which for each object $x^t \in X^t$, $\succeq^t_{x^t} = \geq^t$.

Definition 4. The market-wise serial dictatorship (SD) rule induced by a list of priority orders $\geq = (\geq^t)_{t \in T} \in \prod_{t \in T} \mathcal{P}(N^t)$, denoted by $SD^{\geq}$, selects its type-t allocation as $SD^{\geq^t}(R^t)$ that is chosen by the SD type-rule induced by a type-t priority order.

Remark 8. The serial dictatorship (SD) rule induced by a priority $\geq^* \in \mathcal{P}(N)$ selects an allocation for each profile $R \in \mathcal{D}$ as follows: The highest-priority agent under $\geq^*$ receives her best bundle, the second-highest-priority agent under $\geq^*$ receives her best bundle among remaining objects, and so on.

Note that the SD rule induced by $\geq^*$ coincides with the market-wise SD rule induced by $(\geq^t)_{t \in T}$ when the type-t priority $\geq^t$ is the same as in $\geq |_{N^t}$. In this sense the SD rule is a special case of the market-wise SD rule.

3.4 Market-wise TTC, DA, and SD rules may fail to be Pareto efficient

We use an example to show that market-wise TTC, DA, and SD rules might select a Pareto inefficient allocation.

We consider a simple model with two agents, $N = \{1, 2\}$, and two types of markets, $T = \{1, 2\}$. Suppose that both agents are interested in type-1 and type-2, i.e., $T = T_1 = T_2$. There are two type-1 objects, a and b, and two type-2 objects, c and d, in addition to null objects. All objects are of the unit quota, i.e., $q(a) = q(b) = q(c) = q(d) = 1$. Let $\succeq = (\succeq^t)_{t \in T}$ be the priority profile as described in the following table.

$\succeq^1_a = \succeq^1_b = \succeq^1_{\emptyset^1}$		$\succeq^2_c = \succeq^2_d = \succeq^2_{\emptyset^2}$
1		2
2		1

where for type-1 objects agent 1 has the first priority, while for type-2 objects agent 2 has the first priority. Note that under this priority profile, market-wise TTC, DA, and SD rules coincide. Let $R = (R_1, R_2) \in \mathcal{D}$ be a profile as described in the following table.

Agent 1				Agent 2		
R_1	R_1^1	R_1^2		R_2	R_2^1	R_2^2
(a, c)	$\underline{a}$	c		(a, c)	a	$\underline{c}$
(b, c)	b	$\underline{d}$		(a, d)	$\underline{b}$	d
$\underline{(a, d)}$	$\emptyset^1$	$\emptyset^2$		$\underline{(b, c)}$	$\emptyset^1$	$\emptyset^2$
(b, d)				(b, d)		
$\vdots$				$\vdots$		

Although preferences over bundles are heterogeneous, type preferences are homogeneous where object a is the most preferred to both agents for the type-1 market, and object c is the most preferred to both of them for the type-2 market.

Under the market-wise SD rule, in the type-1 market, as agent 1 has the first priority, agent 1 is assigned object a and agent 2 object b. In the type-2 market, as agent 2 has the first priority, agent 2 is assigned object c and agent 1 object d. That is, agent 1 is assigned bundle (a, d) and agent 2 (b, c). The assigned objects and bundles are underlined in the profile table above.

Consider another allocation where both agents 1 and 2 swap their bundles assigned under the market-wise SD rule and thus agent 1 is now assigned (b, c) and agent 2 (a, d). Clearly this new allocation dominates the allocation under the market-wise SD rule.

Therefore, the above example shows that the market-wise rules of TTC, DA, and SD result in a Pareto inefficient allocation.

4 Main Results

It is known that *strategy-proofness* and *Pareto efficiency* are compatible in our setting (Monte and Tumennasan, 2013). The leading example is a serial dictatorship rule in which agents choose their favorite *bundle* one by one according to a fixed priority order (Remark 8). However, the rule is extremely unfair. For example, when agents have homogeneous preferences and each type object is of unit quotas, the highest priority agent receives her best object for all types which are envied by all of the other agents.

However, as we saw in the previous section, we have a rich class of *strategy-proof* rules each of whose type rules satisfy a weaker efficiency notion of *non-wastefulness*. Because these are not *Pareto efficient* in general, we turn to a weaker notion of efficiency - *second-best efficiency* - rather than *Pareto efficiency*, while focusing on *strategy-proof* rules: a rule φ is **second-best incentive compatible** *if* φ is *strategy-proof* and no *strategy-proof* rule dominates φ. In other words, a *second-best incentive compatible* rule is in the Pareto frontier of the set of *strategy-proof* rules.

We now state the main result of this paper. The proof is in Appendix B.

Theorem 1. *Suppose that a rule φ is independent and for each $t \in T$, the type rule $\Phi^t : \mathcal{P}(X^t)^{N^t} \to \mathcal{A}^t$ is strategy-proof and non-wasteful. Then, φ is second-best incentive compatible.*

In other words, if we adopt a *strategy-proof* and *non-wasteful* rule for each type market, the overall rule is *second-best incentive compatible*. In this sense, Theorem 1 supports both the independent operation of type markets currently done in most real-life markets, and our current practices in market design - independent consideration of the design of each type market.

Since TTC type-rules and DA type-rules are *strategy-proof* and *non-wasteful* (Remarks 4 and 6), we have the following immediate corollaries.

Corollary 1. *For each priority profile $\succeq \in \prod_{t\in T} \mathcal{P}(N^t)^{X^t}$, the market-wise top trading cycles rule $TTC^{\succeq}$ is second-best incentive compatible.*[22]

Corollary 2. *For each priority profile $\succeq \in \prod_{t\in T} \mathcal{P}(N^t)^{X^t}$, the market-wise deferred acceptance rule $DA^{\succeq}$ is second-best incentive compatible.*

To state our next corollary, we introduce a **multiple-type housing market** which is a variant of multiple-type markets.[23] A multiple-type housing market is a multiple-type market $(N, T, (T_i)_{i\in N}, (X^t)_{t\in T}, q, \omega)$ in which the null objects do not necessarily exist. In this paragraph only, we do not assume Assumption 1. Instead we only assume that for each $t \in T$, $\sum_{x^t\in X^t} q(x^t) \geq |N^t|$, i.e., there are enough objects for each agent to receive an object in each market in which she is interested. The allocation $\omega \in \mathcal{A}$ describes the system of property rights in the economy. That is, if ω_i^t is a real object in the type-t market, we interpret that agent i has the property right for the object. Note that the pure distributional case, i.e., no agent has the property right in each market, is a special case of our model. In this case, for each $t \in T$ and each $i \in N^t$, $\omega_i^t = \emptyset^t$. Our model is a generalization of the model in Konishi, Quint, and Wako (2001) and Klaus (2008) in the following four points. (i) Some agents may be interested only in a fraction of T. (ii) For each $t \in T$, $|X^t|$ may not be equal to $|N^t|$. (iii) We do not exclude multiple quota. (iv) We do not exclude indifference in preferences. A **market-wise top trading cycles rule for a multiple-type housing market** is the one induced by a priority profile $\succeq = (\succeq^t)_{t\in T} \in \prod_{t\in T} \mathcal{P}(N^t)^{X^t}$ such that for each type $t \in T$ and each type-t object x^t, if an agent $i \in N^t$ is an owner of x^t, namely $x^t = \omega_i^t$, then for each $j \in N^t$ with $j \succeq_{x^t}^t i$, j is also an owner of x^t. In this setup, we have the following result.

Corollary 3. (Klaus, 2008) *Every market-wise top trading cycles rule for a multiple-type housing market is second-best incentive compatible.*[24]

[22] In a single-type market, a TTC rule is in a subclass of Pápai's (2000) hierarchical exchange rules or Pycia and Ünver's (2009) trading cycles rules that are group strategy-proof and Pareto efficient. Thus, by Theorem 1, if we adopt theirs as type rules, the resulting rule is *second-best incentive compatible*.

[23] This model is sometimes referred to as the generalized Shapley-Scarf housing market. Here, we present a further generalized version of the model in which multiple quotas are allowed.

[24] Rigorously speaking, Corollary 3 is not a direct consequence of Theorem 1 due to Assumption 1. However, our proof immediately implies Corollary 3, which is given in the Appendix.

Our Corollary 3 is more general than Klaus's (2008) original result due to the above four differences in the setup. Furthermore, Theorem 1 is a substantial extension of Klaus's result, as we use *non-wastefulness* for type rules instead of *Pareto efficiency* implied by the TTC type-rule.[25]

If we turn to single-type markets, we have the following corollaries:

Corollary 4. (Kesten, 2010) *In a single-type market, there is no strategy-proof and Pareto efficient rule that dominates the deferred acceptance rule.*[26]

Corollary 5. (Abdulkadiroğlu, Pathak, and Roth, 2009; Erdil, 2011; Kesten and Kurino, 2013). *In a single-type market, the deferred acceptance rule is second-best incentive compatible.*

Corollary 6. (Erdil, 2011) *In a single-type market, a strategy-proof and non-wasteful rule is second-best incentive compatible.*

Corollaries 5 and 6 are respectively counterparts of Corollary 2 and Theorem 1 for a single-type market. However, we cannot show Corollary 2 (Theorem 1) by applying Corollary 5 (Corollary 6) to each type market. To see this, let us recall the proof technique in Abdulkadiroğlu, Pathak, and Roth (2009) and Erdil (2011): Like we do, they suppose for a contradiction that a *strategy-proof* rule, ζ, dominates a *strategy-proof* and *non-wasteful* rule, φ. Then they find some agent i and some profile R with $\zeta_i(R)\ P_i\ \varphi_i(R)\ P_i\ \emptyset$, and then use a special kind of manipulation to upgrade the null object between $\zeta_i(R)$ and $\varphi_i(R)$ in i's preference, which eventually leads to a contradiction. However, this technique does not work in a multiple-type market as the situation is drastically changed in the following sense: A dominating *strategy-proof* rule ζ may assign an object which is worse than the null object in some type markets, because the domination only requires that the **bundle** assigned by ζ is at least as good as the **bundle** assigned by φ. That is, it is possible that $\varphi_i^t(R)\ P_i^t\ \emptyset^t\ P_i^t\ \zeta_i^t(R)$ for some type-t.[27] Note that this situation does not violate $\zeta_i(R)\ R_i\ \varphi_i(R)$ in a multiple-type market. Hence each type-rule in a multiple-type market behaves very differently from a rule in a single-type market.

Before we close this section, let us emphasize the technical advantages of our result. Note that our result is valid in various preference domains including the additively separable weak preference domain. It is known that several technical difficulties arise from the indifference in single-type markets (Erdil and Ergin, 2007). Although type preferences are assumed to be strict, Theorem 1 indicates that the *second-best incentive compatibility* is robust for the indifference with respect to bundles. Furthermore, as pointed out in Kesten and Kurino (2013), the *second-best incentive compatibility* is sensitive to domain restriction. Since the set of additively separable strict preferences is smaller than

[25]To employ *non-wastefulness* for type rules instead of *Pareto efficiency* causes a technical difficulty. A discussion on this point can be found in the Appendix.

[26]We note however that Kesten's (2010) original theorem is also valid when the null object does not exist.

[27]Remember that we assume that φ is independent and each type-rule Φ^t is *strategy-proof* and *non-wasteful* in Theorem 1. Thus, for each $t \in T$, $\varphi_i^t(R)\ R_i\ \emptyset^t$.

the set of all separable strict preferences (See Remark 1), Theorem 1 − more precisely, the preference construction given in Lemma 1 in the Appendix − clarifies a technical limitation of domain restriction for the *second-best incentive compatibility* result.

5 Market Design Applications

We briefly discuss the representative type-rules for the matching problems discussed in the literature. Each subsection looks at a problem with the same notation as described in Section 2.2, and also takes up a *strategy-proof* and at least *non-wasteful* type-rule. Theorem 1 implies that an independent rule of using those type-rules in these type markets is *second-best incentive compatible.*

5.1 House allocation with existing tenants

To remedy the welfare losses observed in practice, inspired by Gale's celebrated assignment method, Abdulkadiroğlu and Sönmez (1999) propose the **top trading cycles (AS-TTC) type-rule** induced by a priority ordering $\geq^t \in \mathcal{P}(N^t)$. The type-rule is the TTC-type rule induced by the following type-t priority profile $\succeq^t$ that is introduced in Section 3.1:[28] For each $x^t \in X^t$, let $N^t(x^t) := \{i \in N^t | \omega_i^t = x^t\}$ be the set of agents who occupy x^t. Note that $N^t(x^t)$ can be empty. Then, for each $x^t \in X^t$, $\succeq^t_{x^t}$ satisfies: (i) $\succeq^t_{x^t} |_{N^t \setminus N^t(x^t)} = \geq^t |_{N^t \setminus N^t(x^t)}$, and (ii) for each $\{i, j\} \subseteq N^t(x^t)$, if $i \in N^t(x^t)$ and $j \succeq^t_{x^t} i$, then $j \in N^t(x^t)$.

Abdulkadiroğlu and Sönmez (1999) show that the AS-TTC type-rule is *strategy-proof* and *Pareto efficient*, which also follows from Remark 4.

5.2 Kidney exchange

Roth, Sönmez, and Ünver (2004) propose an inventory of **top trading cycles and chains** (TTCC) type-rules as a plausible generalization of the top trading cycles method to this setting.[29] A **cycle** is an ordered list $(k_1, i_1, k_2, i_2, \ldots, k_m, i_m)$ such that kidney k_1 points to patient i_1, patient i_1 points to kidney k_2, ..., kidney k_m points to patient i_m, and patient i_m points to kidney k_1. A $\boldsymbol{w}$**-chain** is an ordered list $(k_1, i_1, k_2, i_2, \ldots, k_m, i_m)$ such that kidney k_1 points to patient i_1, patient i_1 points to kidney k_2, ..., kidney k_m points to patient i_m, and patient i_m points to w.

The TTCC algorithm is based on iteratively identifying cycles and w-chains in a directed graph and carrying out the induced trades.[30] The way w-chains are handled in the algorithm depends on

[28]We modify the original description of AS-TTC type rule to use our description of the TTC type-rule in Section 3.1.

[29]In what follows, for breavity we do not provide a self-contained and thorough description of TTCC type-rules. We refer the reader to Roth, Sönmez, and Ünver (2004) for a comprehensive account of the rule. See also Sönmez and Ünver (2013).

[30]Although there are clear similarities between the AS-TTC type-rule in a house allocation problem with existing tenants and the TTCC type-rule in the kidney exchange problem, the adaptations of the top trading cycles method differ in terms of the role the null object plays. In the former context the null object always points to the highest

the so-called *chain selection rule*. Roth, Sönmez, and Ünver (2004) discuss various chain selection rules and investigate their implications for welfare and incentives. Of particular interest to us among these are those rules that induce *strategy-proofness* and *Pareto efficiency* of TTCC.

5.3 School choice

A school choice problem (Abdulkadiroğlu and Sönmez, 2003) is a type market with a type-priority profile. Thus our descritpions of the TTC type-rule and the DA type-rule in Section 3 are for a school choice problem. Thus the TTC type-rule is *strategy-proof* and *Pareto efficient* (Remark 4) and the DA type-rule is *strategy-proof* and *non-wasteful* (Remark 6).

6 Conclusion

In this paper, we consider efficiency for strategy-proof rules in a multiple-type market. The full efficiency is strong enough that we end up with an extremely unfair rule such as the serial dictatorship (Monte and Tumennasan, 2013). We turn to a weaker efficiency notion of the *second-best incentive compatibility* that requires a rule to be *strategy-proof* and not be dominated by any other *strategy-proof* rules. Our main result is the *second-best incentive compatibility* of a market-wise application of *non-wasteful* and *strategy-proof* type rules that include the two well-known priority-based rules of top trading cycles (TTC) and deferred acceptance (DA). This shows that there is a rich class of *second-best incentive compatible* rules, and moreover supports our practices of designing a type rule to be *strategy-proof* and at least *non-wasteful* in Matching Market Design.

We now discuss the existence of the null objects in Assumption 1. As Kesten and Kurino (2013) point out, Corollary 6 does not hold when there is no null object: For example, consider a single-type market with n agents and n objects with unit quotas. Then, a constant allocation rule is *strategy-proof* and *non-wasteful*. However, the rule is dominated by the corresponding core rule which is also *strategy-proof*. With the same logic, our main result of Theorem 1 no longer holds without Assumption 1. As Kesten and Kurino (2013) show, for a single-type market without the null object, the DA rule is *second-best incentive compatible*. It is an interesting open question to prove that for a multiple-type market without null objects, the market-wise DA rule is *second-best incentive compatible*.

In this paper we have focused on deterministic rules. There is a growing literature on lottery rules in matching problems (e.g., Abdulkadiroğlu and Sönmez, 1998; Bogomolnaia and Moulin, 2001; Che and Kojima, 2010). Although the counterpart of Theorem 1 for lottery rules for a single-type market are still true, Erdil (2011) show that the counterpart cannot be applied to an interesting class of lottery rules such as the random serial dictatorship (RSD) which randomly selects a priority and

priority agent, whereas in the latter context, the w-option never points to any agent. This subtle nuance is one main source of the difference between the two rules, and the reason why TTCC type-rules are not described as a special case of the TTC type-rules introduced in Section 3.1.

implements the serial dictatorship for the realized priority. That is, RSD is not *second-best incentive compatible* and thus some *strategy-proof* rule dominates RSD. However, Erdil's rule is quite limited to a small economy and a general dominating rule against stochastic DA or TTC is not known yet. Thus it is an interesting open question as to how much room we can have against those interesting lottery rules for efficiency while keeping *strategy-proofness*. We believe that this paper could be a benchmark in this direction, and could clarify technical limitations and provide technical tools for the question.

A Appendix: Proof of Remark 1

Claim 1. $\mathcal{P}_{lex}(X_i) \subseteq \mathcal{P}_{add}(X_i)$.

Proof. Let $R_i \in \mathcal{P}_{lex}(X_i)$. Suppose that R_i is characterized by $\sigma \in \Sigma(T_i)$ and $(R_i^t)_{t\in T_i} \in \prod_{t\in T_i} \mathcal{P}(X^t)$. Assume, without loss of generality, that $T_i = T$ and σ is the identity mapping, and $(R_i^t)_{t\in T}$ is such that

R_i^1	R_i^2	$\cdots$	$R_i^{\lvert T\rvert}$
x^{11}	x^{21}		$x^{\lvert T\rvert 1}$
x^{12}	x^{22}		$x^{\lvert T\rvert 2}$
$\vdots$	$\vdots$	$\cdots$	$\vdots$
x^{1K_1}	x^{2K_2}		$x^{\lvert T\rvert K_{\lvert T\rvert}}$

where for each $t \in T$, $K_t := |X^t|$. Let $K := \sum_{t\in T} K_t$. Now we define $(u_i^t)_{t\in T} \in \Pi_{t\in T}\mathcal{U}^t$ as follows:

$$\begin{array}{cccc}
u_i^1(x^{11}) = 10^{K-1} & u_i^2(x^{21}) = 10^{K-K_1-1} & & u_i^{|T|}(x^{|T|1}) = 10^{K-\sum_{t=1}^{|T|-1} K_t-1} \\
u_i^1(x^{12}) = 10^{K-2} & u_i^2(x^{22}) = 10^{K-K_1-2} & & u_i^{|T|}(x^{|T|2}) = 10^{K-\sum_{t=1}^{|T|-1} K_t-2} \\
\vdots & \vdots & \cdots & \vdots \\
u_i^1(x^{1K_1}) = 10^{K-K_1} & u_i^2(x^{2K_2}) = 10^{K-(K_1+K_2)} & & u_i^{|T|}(x^{|T|K_{|T|}}) = 10^{K-\sum_{t=1}^{|T|} K_t}.
\end{array}$$

Obviously, for each $\{y_i, z_i\} \subseteq X_i$, $y_i\ R_i\ z_i$ *if and only if* $\sum_{t\in T} u_i^t(y_i^t) \geq \sum_{t\in T} u_i^t(z_i^t)$. Thus $R_i \in \mathcal{P}_{add}(X_i)$. □

Claim 2. $\mathcal{P}_{add}(X_i) \neq \mathcal{P}_{sep}(X_i)$.

Proof. The proof idea is similar to the one in page 43 in Fishburn (1970) that shows $\mathcal{R}_{add}(X_i) \subsetneq \mathcal{R}_{sep}(X_i)$. Suppose that $X^1 = X^2 = \{x, y, z\}$ and $X_i = X^1 \times X^2$. Let R_i be the preference such that

$$(x, x)\ P_i\ (x, y)\ P_i\ (y, x)\ P_i\ (z, x)\ P_i\ (y, y)\ P_i\ (x, z)\ P_i\ (y, z)\ P_i\ (z, y)\ P_i\ (z, z).$$

Obviously $R_i \in \mathcal{P}_{sep}(X_i)$. We show $R_i \notin \mathcal{P}_{add}(X_i)$. Suppose to the contrary that $(u^1, u^2) \in \mathcal{U}^1 \times \mathcal{U}^2$ represents R_i. Since $(z, x)\ P_i\ (x, z)$ and $(y, z)\ P_i\ (z, y)$, $u^1(z)+u^2(x) > u^1(x)+u^2(z)$ and $u^1(y)+u^2(z) > u^1(z) + u^2(y)$. Thus $u^1(z) + u^2(x) + u^1(y) + u^2(z) > u^1(x) + u^2(z) + u^1(z) + u^2(y)$. Cancelling out $u^1(z)$ and $u^2(z)$, we obtain $u^1(y) + u^2(x) > u^1(x) + u^2(y)$. This violates that $(x, y)\ P_i\ (y, x)$. □

B Appendix: Proof of Theorem 1

We first introduce some notations: For each $\succsim \in \mathcal{R}(Y)$ and each $y \in Y$, let UC$(\succsim, y)$, SUC$(\succsim, y)$, LC$(\succsim, y)$ and SLC$(\succsim, y)$ be the upper, strict upper, lower, and strict lower contour set of $\succsim$ at y, respectively. That is, UC$(\succsim, y) := \{z \in Y | z \succsim y\}$, SUC$(\succsim, y) := \{z \in Y | z \succsim y$ and not $y \succsim z\}$, LC$(\succsim, y) := \{z \in Y | y \succsim z\}$ and SLC$(\succsim, y) := \{z \in Y | y \succsim z$ and not $z \succsim y\}$.

Before we prove Theorem 1, we provide four lemmas. Lemma 1 states that the domain $\mathcal{D}$ is rich enough to choose the preferences we need in the proofs of subsequent lemmas and theorem.

Lemma 1. *Let $i \in N$. Let $(\tilde{R}_i^t) \in \prod_{t \in T_i} \mathcal{P}(X^t)$ and $\tilde{x}_i \in X_i$. There exists $R_i \in \mathcal{P}_{add}(X_i)$ such that*

(i) $\forall t \in T_i, R_i^t = \tilde{R}_i^t$ *and*

(ii) $\forall x_i \in X_i, [\{\exists t \in T_i \text{ s.t. } x_i^t \in SLC(\tilde{R}_i^t, \tilde{x}_i^t)\} \Rightarrow \tilde{x}_i \; P_i \; x_i]$.

Proof. Without loss of generality, suppose that $T_i = T$. For each $t \in T$, let $X^t = \{x^{t1}, \cdots, x^{t|X^t|}\}$, and assume, without loss of generality, that $x^{t1} \; \tilde{R}_i^t \; x^{t2} \; \tilde{R}_i^t \cdots \tilde{R}_i^t \; x^{t|X^t|}$. For each $t \in T$, let $k_t \in \mathbb{Z}_{++}$ be the cardinality of UC$(\tilde{R}_i^t, \tilde{x}_i^t)$ where $x^{tk_t} = \tilde{x}_i^t$. Let $k'_t := |X^t| - k_t$, $K := \sum_{t \in T} k_t$ and $K' := \sum_{t \in T} k'_t$. Define $(u_i^t) \in \prod_{t \in T} \mathcal{U}^t$ as follows:

$$
\begin{array}{cccc}
u_i^1(x^{11}) = 10^{K-1}, & u_i^1(x^{12}) = 10^{K-2}, & \cdots & u_i^1(x^{1k_1}) = 10^{K-k_1}, \\
u_i^2(x^{21}) = 10^{K-(k_1+1)}, & u_i^2(x^{22}) = 10^{K-(k_1+2)}, & \cdots & u_i^2(x^{2k_2}) = 10^{K-(k_1+k_2)}, \\
\vdots & \vdots & & \vdots \\
u_i^{|T|}(x^{|T|1}) = 10^{K-(\sum_{t=1}^{|T|-1} k_t+1)}, & u_i^{|T|}(x^{|T|2}) = 10^{K-(\sum_{t=1}^{|T|-1} k_t+2)}, & \cdots & u_i^{|T|}(x^{|T|k_{|T|}}) = 10^{K-(\sum_{t\in T} k_t)} (= 10^0).
\end{array}
$$

Let $C' := 2\sum_{t \in T} u_i^t(x^{t1})$ and $C := C' + 1$.[31] In the following, for each $t \in T$, if $k'_t = 0$, then the corresponding row should be skipped.

$$
\begin{array}{cccc}
u_i^1(x^{1(k_1+1)}) = \frac{1}{10} - C, & u_i^1(x^{1(k_1+2)}) = \frac{1}{10^2} - C, & \cdots & u_i^1(x^{1|X^1|}) = \frac{1}{10^{k'_1}} - C, \\
u_i^2(x^{2(k_2+1)}) = \frac{1}{10^{k'_1+1}} - C, & u_i^2(x^{2(k_2+2)}) = \frac{1}{10^{k'_1+2}} - C, & \cdots & u_i^2(x^{2|X^2|}) = \frac{1}{10^{k'_1+k'_2}} - C, \\
\vdots & \vdots & & \vdots \\
u_i^{|T|}(x^{|T|(k_{|T|}+1)}) = \frac{1}{10^{\sum_{t=1}^{|T|-1} k'_t+1}} - C, & u_i^{|T|}(x^{|T|(k'_{|T|}+1)}) = \frac{1}{10^{\sum_{t=1}^{|T|-1} k'_t+2}} - C, & \cdots & u_i^{|T|}(x^{|T||X^{|T|}|}) = \frac{1}{10^{\sum_{t=1}^{|T|} k'_t}} - C.
\end{array}
$$

Now, we define $R_i \in \mathcal{R}(X_i)$ as follows: for each $\{y_i, z_i\} \subseteq X_i$,

$$y_i \; R_i \; z_i \Leftrightarrow \sum_{t \in T} u_i^t(y_i^t) \geq \sum_{t \in T} u_i^t(z_i^t).$$

Obviously, $R_i \in \mathcal{R}_{add}(X_i) \subseteq \mathcal{R}_{sep}(X_i)$. It is also obvious that $(R_i^t)_{t \in T} = (\tilde{R}_i^t)_{t \in T}$, i.e., Item (i) is satisfied.

[31] Note that C' is greater than the utility obtained by the bundle $(x^{11}, \cdots, x^{|T|1})$ which is the best bundle according to the resulting preference R_i.

We show that R_i satisfies Item (ii). Let $y_i \in X_i$ be such that for some $t \in T$, $y_i^t \in \text{SLC}(R_i^t, \tilde{x}_i^t)$. Note that y_i has at least one coordinate whose utility contains the $-C$ term while $\tilde{x}_i$ does not. Therefore, $\sum_{t\in T} u_i^t(\tilde{x}_i^t) > 0 > \sum_{t\in T} u_i^t(y_i^t)$. Thus, $\tilde{x}_i \ P_i \ y_i$. Thus, Item (ii) is satisfied.

Finally, we prove that $R_i \in \mathcal{P}_{add}(X_i)$. By construction, $R_i \in \mathcal{R}_{add}(X_i)$. Since $\mathcal{P}_{add}(X_i) = \mathcal{R}_{add}(X_i) \cap \mathcal{P}(X_i)$, we need to show $R_i \in \mathcal{P}(X_i)$. Let y_i, z_i be such that $\sum_{t\in T} u_i^t(y_i^t) = \sum_{t\in T} u_i^t(z_i^t)$. First, we claim that the number of types in which the type object is worse than $\tilde{x}_i^t$ is the same between y_i and z_i.

Claim 3. $|\{t \in T | \tilde{x}_i^t \ P_i^t \ y_i^t\}| = |\{t \in T | \tilde{x}_i^t \ P_i^t \ z_i^t\}|$.

Let $\alpha := |\{t \in T | \tilde{x}_i^t \ P_i^t \ y_i^t\}|$ and $\beta := |\{t \in T | \tilde{x}_i^t \ P_i^t \ z_i^t\}|$. Suppose to the contrary that $\alpha \neq \beta$. Assume, without loss of generality, that $\alpha < \beta$. We can decompose the utility into three parts:

$$\sum_{t\in T} u_i^t(y_i^t) = \sum_{\substack{t\in T \\ y_i^t R_i^t \tilde{x}_i^t}} u_i^t(y_i^t) + (\sum_{\substack{t\in T \\ \tilde{x}_i^t P_i^t y_i^t}} u_i^t(y_i^t) + \alpha C) - \alpha C.$$

Since $0 \leq (\sum_{\substack{t\in T \\ \tilde{x}_i^t P_i^t y_i^t}} u_i^t(y_i^t) + \alpha C) < 1$, we have

$$\sum_{\substack{t\in T \\ y_i^t R_i^t \tilde{x}_i^t}} u_i^t(y_i^t) - \alpha C \leq \sum_{t\in T} u_i^t(y_i^t) \leq \sum_{\substack{t\in T \\ y_i^t R_i^t \tilde{x}_i^t}} u_i^t(y_i^t) + 1 - \alpha C.$$

Similarly, we have

$$\sum_{\substack{t\in T \\ z_i^t R_i^t \tilde{x}_i^t}} u_i^t(z_i^t) - \beta C \leq \sum_{t\in T} u_i^t(z_i^t) \leq \sum_{\substack{t\in T \\ z_i^t R_i^t \tilde{x}_i^t}} u_i^t(z_i^t) + 1 - \beta C.$$

Thus, we have

$$\begin{aligned}
&(\sum_{\substack{t\in T \\ y_i^t R_i^t \tilde{x}_i^t}} u_i^t(y_i^t) - \alpha C) - (\sum_{\substack{t\in T \\ z_i^t R_i^t \tilde{x}_i^t}} u_i^t(z_i^t) + 1 - \beta C) = -1 + (\sum_{\substack{t\in T \\ y_i^t R_i^t \tilde{x}_i^t}} u_i^t(y_i^t) - \sum_{\substack{t\in T \\ z_i^t R_i^t \tilde{x}_i^t}} u_i^t(z_i^t)) + (\beta - \alpha)C \\
&\geq -1 + (-\tfrac{1}{2}C') + (\beta - \alpha)C \qquad \left(\because \sum_{\substack{t\in T \\ y_i^t R_i^t \tilde{x}_i^t}} u_i^t(y_i^t) - \sum_{\substack{t\in T \\ z_i^t R_i^t \tilde{x}_i^t}} u_i^t(z_i^t) \geq -\frac{1}{2}C'\right) \\
&= -1 + (-\tfrac{1}{2}C') + (\beta - \alpha)C' + (\beta - \alpha) \geq \tfrac{1}{2}C' > 0. \qquad (\because \ \alpha < \beta)
\end{aligned}$$

Thus, $\sum_{t\in T} u_i^t(z_i^t) < \sum_{t\in T} u_i^t(y_i^t)$, a contradiction. Thus, $\alpha = \beta$. The proof of the Claim is completed.

Now we complete the proof of $R_i \in \mathcal{P}(X_i)$. Let us express $Y := \sum_{t\in T} u_i^t(y_i^t) + \alpha C$ as a $(K + K')$ digits rational number. That is, $Y = Y_1 Y_2 \cdots Y_K . Y_{K+1} Y_{K+2} \cdots Y_{K+K'}$.[32] Note that by the construction, each digit is equal to 1 or 0. Note also that $(Y_1, \cdots, Y_{k_1}, Y_{K+1}, \cdots, Y_{K+k_1'})$ tells us which type-1

[32]Note that Y_1 denotes the 10^{K-1}'s place of Y (which may be 0), Y_2 denotes the 10^{K-2}'s place of Y (which may be 0) and so on. Similarly, Y_{K+1} denotes the $\frac{1}{10}$'s place of Y, Y_{K+2} denotes the $\frac{1}{10^2}$'s place of Y and so on.

object is assigned at y_i since for each $k \in \{1, \cdots k_1, K+1, \cdots, K+k'_1\}$, $Y_k = 1$ *if and only if*

$$y_i^1 = \begin{cases} x^{1k} & \text{if } 1 \leq k \leq k_1 \\ x^{1(k-K+k_1)} & \text{otherwise} \end{cases}$$

In general, for $t \geq 2$, $(Y_{\sum_{t'<t} k_{t'}+1}, \cdots, Y_{\sum_{t' \geq t} k_{t'}}, Y_{K+\sum_{t'<t} k'_{t'}+1}, \cdots, Y_{K+\sum_{t' \geq t} k'_{t'}})$ tells us the type-t object at y_i. Therefore, we can identify the bundle y_i with the value of Y. Similarly, let $Z := \sum_{t \in T} u_i^t(z_i^t) + \beta C$ and

$$Z = Z_1 Z_2 \cdots Z_K . Z_{K+1} Z_{K+2} \cdots Z_{K+K'}.$$

Since $\sum_{t \in T} u_i^t(y_i^t) = \sum_{t \in T} u_i^t(z_i^t)$ and $\alpha = \beta$ ($\because$ Claim), we have $Y = \sum_{t \in T} u_i^t(y_i^t) + \alpha C = \sum_{t \in T} u_i^t(z_i^t) + \beta C = Z$. Thus, $y_i = z_i$. $\square$

We introduce notations: For each $i \in N$, each $t \in T_i$, each $R_i^t \in \mathcal{P}(X^t)$, each $R_i \in \mathcal{D}_i$ and each $R \in \mathcal{D}$, let

$$\begin{aligned} B(R_i^t) &:= |\text{SUC}(R_i^t, \emptyset^t)|, \\ B(R_i) &:= \sum_{t \in T_i} B(R_i^t), \text{ and} \\ B(R) &:= \sum_{i \in N} B(R_i). \end{aligned}$$

Namely, B is the operator that assigns the number of object(s) which are preferred to the null object(s). For each $R \in \mathcal{D}$, let $I(R)$ be the number of agents whose preferences are not strict at R, i.e.,

$$I(R) := |\{i \in N | R_i \notin \mathcal{P}(X_i)\}|.$$

The following two notions are the key to the proof of Theorem 1. Given $R \in \mathcal{D}$, an allocation $a \in \mathcal{A}$ **coordinate-wise weakly dominates** $b \in \mathcal{A}$ at R, written as a cw-dom(R) b, *if*

$$\forall i \in N, \forall t \in T_i, a_i^t \; R_i^t \; b_i^t.$$

Given a pair of rules (ζ, φ), a profile $R \in \mathcal{D}$ satisfies the $(\boldsymbol{\zeta}, \boldsymbol{\varphi})$**−reverse property** *if*

$$\exists i \in N \text{ s.t. } \Big[\zeta_i(R) \; P_i \; \varphi_i(R) \text{ and } \{\exists t \in T_i \text{ s.t. } \varphi_i^t(R) \; P_i^t \; \zeta_i^t(R)\}\Big].^{33}$$

Let us sketch the proof of Theorem 1.[34] The proof shall be done by a contradiction. Therefore, we assume that there exists a *strategy-proof* rule ζ which dominates φ. Let $R^{(1)} \in \mathcal{D}$ be such that $\zeta(R^{(1)})$ dominates $\varphi(R^{(1)})$ at $R^{(1)}$. Starting from this, we inductively construct two sequences of non-negative integers $\{N^{(k)}\}_{k=0}^{\infty}$ and $\{B^{(k)}\}_{k=1}^{\infty}$ satisfying Items (seq-i) and (seq-ii).

[33] Note that *if* ζ dom φ and $R \in \mathcal{D}$ satisfies (ζ, φ)−reverse property, then $\zeta(R)$ dom(R) $\varphi(R)$.

[34] Our proof is greatly inspired by the one in Klaus (2008).

(seq-i) $\{N^{(k)}\}_{k=0}^{\infty}$ is weakly decreasing, i.e., $N^{(0)} \geq N^{(1)} \geq N^{(2)} \geq \cdots$.

(seq-ii) If $N^{(k-1)}$ does not decrease ($N^{(k-1)} = N^{(k)}$), then the corresponding part of $\{B^{(k)}\}_{k=1}^{\infty}$ decreases, i.e., for each $k \in \mathbb{N}$, if $N^{(k-1)} = N^{(k)}$, then $B^{(k)} > B^{(k+1)}$.[35]

In each induction step of the proof, we shall choose an agent $i^{(k)}$ and her new preference $R_{i^{(k)}}^{(k+1)}$, and define $N^{(k)} := |N \backslash \{i^{(1)}, \cdots, i^{(k)}\}|$ and $B^{(k+1)} := B(R_{i^{(k)}}^{(k+1)}; R_{-i^{(k)}}^{(k)})$. It is a process of successive preference replacements to satisfy Item (seq-ii) as Item (i) is automatically satisfied by definition, which is made possible by the (ζ, φ)-reverse property at the profile under consideration.[36] The simplest case is when every profile satisfies the reverse property. One such simple case is: each type-rule Φ^t is *Pareto efficient* and $\mathcal{D} = \prod_{i \in N} \mathcal{P}_{sep}(X_i)$.[37]

However, the assumptions of Theorem 1 navigate away the above situation in two ways. First, we only assume *non-wastefulness* for each type rule instead of *Pareto efficiency*. Second, the preference domain contains weak preferences. Even under one of these weak assumptions, it is easy to show that a given preference profile may not satisfy the (ζ, φ)-reverse property. For this reason, in addition to the process constructing $\{N^{(k)}\}_{k=0}^{\infty}$ and $\{B^{(k)}\}_{k=1}^{\infty}$ (let us call it the constructing process), we need another process to transit from a profile without the (ζ, φ)-reverse property toward a profile with the (ζ, φ)-reverse property (let us call it the transition process). Lemmas 2 to 4 show how we transit from one profile to another. Under a given profile without the (ζ, φ)-reverse property, we use two types of transition according to whether the coordinate-wise weak domination occurs in the profile or not. Lemma 2 describes the transition when the domination occurs, while Lemma 3 describes when the domination does not. Lemma 4 guarantees that repeating these transitions, we finally reach a profile with the (ζ, φ)-reverse property. In sum, the proof of Theorem 1 is the constructing process in which each step contains the transition process.[38]

Note that both the constructing process and the transition process contain preference replacements of agents. When we prove that the resulting sequences satisfy (seq-ii), we need the following: once an agent is involved in the preference replacement with respect to the constructing process, she is never involved in the preference replacement with respect to the subsequent transition process. In other words, we need to design the transition process so as not to disturb the constructing process. This trick is realized by Items (lem2-1), (lem3-1) and Item (i) in Lemma 4.

Now, we present Lemma 2. It tells us how we transit from a profile *if* the coordinate-wise weak domination occurs in the profile. Item (lem2-2) is needed to terminate the induction process in Lemma 4 in a finite number of steps, and it is also needed to prove Item (ii) in Lemma 4, which is also a trick

[35] Note that these sequences cause a contradiction since sequences of non-negative integers cannot meet both (seq-i) and (seq-ii).

[36] The detailed construction of a new preference is indicated by conditions $(1^* - i)$, $(1^* - ii)$, $(k^* - i)$ and $(k^* - ii)$ in the proof of Theorem 1.

[37] See Lemma 1 in Klaus (2008).

[38] Therefore, the profiles in the proof of Theorem 1 are doubly indexed. The first index indicates the steps of the constructing process, and the second index indicates the steps of the transition process.

to prove that two sequences in the proof of Theorem 1 satisfies (seq-ii). Item (lem2-3) guarantees that the induction argument in Lemma 4 bites.

Lemma 2. *Suppose that a rule φ is independent and for each $t \in T$, $\Phi^t : \mathcal{P}(X^t)^{N^t} \to \mathcal{A}^t$ is strategy-proof and non-wasteful. Suppose also that a strategy-proof rule ζ dominates φ. Let $R \in \mathcal{D}$ be such that $\zeta(R)$ dom(R) $\varphi(R)$. If $\zeta(R)$ cw-dom(R) $\varphi(R)$, then there exist $i \in N$ and $R'_i \in \mathcal{D}_i$ such that*

(lem2-1) $\exists\, t \in T_i$ *s.t.* $\zeta_i^t(R)\ P_i^t\ \varphi_i^t(R)\ P_i^t\ \emptyset^t$,

(lem2-2) $B(R'_i; R_{-i}) < B(R)$ *and* $I(R'_i; R_{-i}) \leq I(R)$, *and*

(lem2-3) $\zeta(R'_i; R_{-i})$ *dom*$(R'_i; R_{-i})$ $\varphi(R'_i; R_{-i})$.

Proof. First, we show that there exists $i \in N$ satisfying (lem2-1). Suppose to the contrary that for each $i \in N$ and each $t \in T_i$, $[\varphi_i^t(R)\ P_i^t\ \emptyset^t \Rightarrow \varphi_i^t(R)\ R_i^t\ \zeta_i^t(R)]$. Since $\zeta(R)$ cw-dom(R) $\varphi(R)$, the hypothesis is equivalent to

$$\forall i \in N, \forall t \in T_i, [\varphi_i^t(R)\ P_i^t\ \emptyset^t \Rightarrow \varphi_i^t(R) = \zeta_i^t(R)]. \tag{1}$$

Claim 4. $\forall i \in N, \forall t \in T_i, [\varphi_i^t(R) = \emptyset^t \Rightarrow \zeta_i^t(R) = \emptyset^t]$.
Suppose not. Let $i \in N$ and $t \in T_i$ be such that $\varphi_i^t(R) = \emptyset^t$ and $\zeta_i^t(R) \neq \emptyset^t$. Let $x^t := \zeta_i^t(R)$ and $a^t := (\varphi_i^t(R))_{i \in N^t}$. We show that at least one unit of the real object x^t is not assigned to any agent at the type-t allocation a^t. By (1), for each $j \in N^t$, $[a_j^t = x^t \Rightarrow \zeta_j^t(R) = x^t]$. Thus, since $a_i^t = \emptyset^t$ and $\zeta_i^t(R) = x^t$, we have $|\{j \in N^t | a_j^t = x^t\}| < |\{j \in N^t | \zeta_j^t(R) = x^t\}| \leq q(x^t)$. However, since $\zeta(R)$ cw-dom(R) $\varphi(R)$ and type preferences are strict, $x^t = \zeta_i^t(R)\ P_i^t\ \varphi_i^t(R) = \Phi_i^t(R^t)$, a violation of *non-wastefulness* of Φ^t. This completes the proof of the Claim.

Note that by Remark 2, Φ^t is *individually rational.* Thus, by (1) and the Claim, for each $i \in N$ and each $t \in T_i$, $\varphi_i^t(R) = \zeta_i^t(R)$. Thus, $\varphi(R) = \zeta(R)$, which violates our assumption that $\zeta(R)$ dom(R) $\varphi(R)$. Therefore, there exists $i \in N$ satisfying (lem2-1).

Next, let $i \in N$ and $t_0 \in T_i$ be such that $\zeta_i^{t_0}(R)\ P_i^{t_0}\ \varphi_i^{t_0}(R)\ P_i^{t_0}\ \emptyset^{t_0}$. We show that there exists a preference $R'_i \in \mathcal{D}_i$ which satisfies (lem2-2). First, we define a list of type preferences $(\tilde{R}_i'^t)_{t \in T_i}$. In words, $(\tilde{R}_i'^t)_{t \in T_i}$ is obtained from $(R_i^t)_{t \in T_i}$ by changing the ranking of $\emptyset^{t_0}$ just above $\varphi_i^{t_0}(R)$ while the relative rankings of any other objects stay the same. Formally, $(\tilde{R}_i'^t)_{t \in T_i}$ is defined as follows;

- $[\forall t \in T_i \backslash \{t_0\}, \tilde{R}_i'^t = R_i^t]$ and $\tilde{R}_i'^{t_0}|_{(X^{t_0} \backslash \{\emptyset^{t_0}\}) \times (X^{t_0} \backslash \{\emptyset^{t_0}\})} = R_i^{t_0}|_{(X^{t_0} \backslash \{\emptyset^{t_0}\}) \times (X^{t_0} \backslash \{\emptyset^{t_0}\})}$,
- $\emptyset^{t_0}\ \tilde{P}_i'^{t_0}\ \varphi_i^{t_0}(R)$ and
- $\nexists x^{t_0} \in X^{t_0} \backslash \{\emptyset^{t_0}, \varphi_i^{t_0}(R)\}$ s.t. $\emptyset^{t_0}\ \tilde{R}_i'^{t_0}\ x^{t_0}\ \tilde{R}_i'^{t_0}\ \varphi_i^{t_0}(R)$.

By applying Lemma 1 for $(\tilde{R}_i'^t)_{t \in T_i}$ and $\zeta_i(R)$, we can choose a preference $R'_i \in \mathcal{P}_{add}(X_i)$ such that

$$\forall x_i \in X_i, [\{\exists t \in T_i \text{ s.t. } x_i^t \in \text{SLC}(R_i'^t, \zeta_i^t(R))\} \Rightarrow \zeta_i(R)\ P'_i\ x_i]. \tag{2}$$

By (lem2-1) and the construction of $(R_i'^t)_{t \in T_i}$, $B(R_i') < B(R_i)$. Therefore, $B(R_i'; R_{-i}) < B(R)$. Since $R_i' \in \mathcal{P}_{add}(X_i)$, $I(R_i'; R_{-i}) \leq I(R)$.

Finally, we show (lem2-3). Since Φ^{t_0} is *strategy-proof* and *individually rational*, $\Phi_i^{t_0}(R_i'^{t_0}; R_{-i}^{t_0}) = \emptyset^{t_0}$. Note that since $\emptyset^{t_0} \in \mathrm{SLC}(R_i'^{t_0}, \zeta_i^{t_0}(R))$, by (2), we have $\zeta_i(R)\ P_i'\ \varphi_i(R_i'; R_{-i})$. Thus, by *strategy-proofness* of ζ, $\zeta_i(R_i'; R_{-i})\ P_i'\ \varphi_i(R_i'; R_{-i})$. Since ζ dom φ, we are done. □

Next we present Lemma 3. It tells us how we transit from a profile *if* the coordinate-wise weak domination does not occur in the profile. Item (lem3-2) is needed to terminate the induction process in Lemma 4 in a finite number of steps, and it is also needed to prove Item (ii) in Lemma 4, which is also a trick to prove that two sequences in the proof of Theorem 1 satisfies (seq-ii). Item (lem3-3) guarantees that the induction argument in Lemma 4 bites.

Lemma 3. *Suppose that a rule φ is independent and for each $t \in T$, $\Phi^t : \mathcal{P}(X^t)^{N^t} \to \mathcal{A}^t$ is strategy-proof. Suppose also that a strategy-proof rule ζ dominates φ. Let $R \in \mathcal{D}$. If R does not satisfy the (ζ, φ)-reverse property, and not $\zeta(R)$ cw-dom(R) $\varphi(R)$, then there exist $i \in N$ and $R_i' \in \mathcal{D}_i$ such that*

(lem3-1) *$R_i \notin \mathcal{P}(X_i)$,*

(lem3-2) *$I(R_i'; R_{-i}) < I(R)$ and $B(R_i'; R_{-i}) \leq B(R)$, and*

(lem3-3) *$\zeta(R_i'; R_{-i})$ dom$(R_i'; R_{-i})$ $\varphi(R_i'; R_{-i})$.*

Proof. First, we show that there exists $i \in N$ satisfying (lem3-1). Since not $\zeta(R)$ cw-dom(R) $\varphi(R)$, there exist $i \in N$ and $t_0 \in T_i$ such that $\varphi_i^{t_0}(R)\ P_i^{t_0}\ \zeta_i^{t_0}(R)$. Thus, since R does not satisfy the (ζ, φ)-reverse property, we have $\varphi_i(R)\ R_i\ \zeta_i(R)$. Since ζ dom φ, $\varphi_i(R)\ I_i\ \zeta_i(R)$. Since $\varphi_i(R) \neq \zeta_i(R)$, we obtain that $R_i \notin \mathcal{P}(X_i)$, i.e., (lem3-1).

Next we choose $R_i' \in \mathcal{D}_i$ as follows. By Assumption 2, $R_i \in R_{add}\ (X_i)$. Let $(u_i^t)_{t \in T_i} \in \prod_{t \in T_i} \mathcal{U}^t$ be a list of type-utility functions which represents R_i. Since $u_i^{t_0}(\varphi_i^{t_0}(R)) > u_i^{t_0}(\zeta_i^{t_0}(R))$ and $\sum_{t \in T_i} u_i^t(\varphi_i^t(R)) = \sum_{t \in T_i} u_i^t(\zeta_i^t(R))$, there exists $t_1 \in T_i$ such that $u_i^{t_1}(\zeta_i^{t_1}(R)) > u_i^{t_1}(\varphi_i^{t_1}(R))$, i.e., $\zeta_i^{t_1}(R)\ P_i^{t_1}\ \varphi_i^{t_1}(R)$. Now we change the preference of agent i. Let $(R_i'^t)_{t \in T_i} := (R_i^t)_{t \in T_i}$ and $\sigma \in \Sigma(T_i)$ be such that $\sigma(1) = t_1$. Let $R_i' \in \mathcal{P}_{lex}(X_i)$ be the lexicographic preference defined by σ and $(R_i'^t)_{t \in T_i}$ (See Remark 1). Obviously, $I(R_i'; R_{-i}) < I(R)$ and $B(R_i'; R_{-i}) = B(R)$. Thus, Item (3-2) is satisfied.

Finally we show (lem3-3). First, since φ is independent and $R_i'^{t_1} = R_i^{t_1}$, $\varphi_i^{t_1}(R_i'; R_{-i}) = \varphi_i^{t_1}(R)$. Second, since ζ is *strategy-proof*, $\zeta_i(R_i'; R_{-i})\ R_i'\ \zeta_i(R)$. Third, since R_i' is a lexicographic preference whose first priority is assigned to t_1, we have $\zeta_i^{t_1}(R_i'; R_{-i})\ R_i'^{t_1}\ \zeta_i^{t_1}(R)$. Therefore, since $\zeta_i^{t_1}(R)\ P_i^{t_1}\ \varphi_i^{t_1}(R)$ and $R_i'^{t_1} = R_i^{t_1}$, we have $\zeta_i^{t_1}(R_i'; R_{-i})\ R_i'^{t_1}\ \zeta_i^{t_1}(R)\ P_i'^{t_1}\ \varphi_i^{t_1}(R) = \varphi_i^{t_1}(R_i'; R_{-i})$. Since R_i' is a lexicographic preference whose first priority is on t_1, $\zeta_i(R_i'; R_{-i})\ P_i'\ \varphi_i(R_i'; R_{-i})$. Since ζ dom φ, we are done. □

The following lemma is the realization of the transition process we employ. Item (i) is the trick we need when we prove the two resulting sequences of non-negative integers in the proof of Theorem

1 meet the condition (seq-ii). Item (ii) is also a trick used to prove the condition (seq-ii). Item (iii) asserts that the transition process finally reaches a profile which satisfies the (ζ,φ)-reverse property.

Lemma 4. *Suppose that a rule φ is independent and for each $t \in T$, $\Phi^t : \mathcal{P}(X^t)^{N^t} \to \mathcal{A}^t$ is strategy-proof and non-wasteful. Suppose also that a strategy-proof rule ζ dominates φ. Let $R^{(0)} \in \mathcal{D}$ be such that $\zeta(R^{(0)})$ dom(R^0) $\varphi(R^{(0)})$. If $R^{(0)}$ does not satisfy the (ζ,φ)-reverse property, then there exists a finite sequence of agent-preference pairs $\{(j^{(\ell)}, R^{(\ell)}_{j^{(\ell)}})\}_{\ell=1}^{L}$ satisfying the following conditions (i), (ii) and (iii). For each $\ell = 1, \cdots, L$, let $R^{(\ell)} := (R^{(\ell)}_{j^{(\ell)}}; R^{(\ell-1)}_{-j^{(\ell)}})$.*

(i) $\forall \ell = 1, \cdots, L$, $\left[\{\exists t \in T_{j^{(\ell)}} \text{ s.t. } |SUC(R^{(\ell-1)t}_{j^{(\ell)}}, \emptyset^t)| \geq 2\} \text{ or } R^{(\ell-1)}_{j^{(\ell)}} \notin \mathcal{P}(X_{j^{(\ell)}})\right]$

(ii) $B(R^{(0)}) \geq B(R^{(1)}) \geq \cdots \geq B(R^{(L)})$, *and*

(iii) $R^{(L)}$ *satisfies the (ζ,φ)-reverse property.*

Proof. We inductively construct a sequence.
Step 1: If $\zeta(R^{(0)})$ cw-dom$(R^{(0)})$ $\varphi(R^{(0)})$, then by Lemma 2, there exist $j^{(1)} \in N$ and $R^{(1)}_{j^{(1)}} \in \mathcal{D}_{j^{(1)}}$ satisfying (lem2-1), (lem2-2) and (lem2-3). If not, then by Lemma 3, there exist $j^{(1)} \in N$ and $R^{(1)}_{j^{(1)}} \in \mathcal{D}_{j^{(1)}}$ satisfying (lem3-1), (lem3-2) and (lem3-3). Then, by (lem2-1) or (lem3-1), {there is $t \in T_{j^{(1)}}$ s.t. $|\text{SUC}(R^{(0)t}_{j^{(1)}}, \emptyset^t)| \geq 2\}$ or $R^{(0)}_{j^{(1)}} \notin \mathcal{P}(X_{j^{(1)}})$. Moreover, by (lem2-2) or (lem3-2), $B(R^{(0)}) \geq B(R^{(1)})$. Therefore, if $R^{(1)}$ satisfies the (ζ,φ)-reverse property, then $(j^{(1)}, R^{(1)}_{j^{(1)}})$ is the desired sequence with its length 1. If $R^{(1)}$ does not satisfy the (ζ,φ)-reverse property, then go to the next step. Note that by (lem2-3) or (lem3-3), $\zeta(R^{(1)})$ dom$(R^{(1)})$ $\varphi(R^{(1)})$.[39]
Step ℓ (≥ 2): If $\zeta(R^{(\ell-1)})$ cw-dom$(R^{(\ell-1)})$ $\varphi(R^{(\ell-1)})$, then by Lemma 2, there exist $j^{(\ell)} \in N$ and $R^{(\ell)}_{j^{(\ell)}} \in \mathcal{D}_{j^{(\ell)}}$ satisfying (lem2-1), (lem2-2) and (lem2-3). If not, then by Lemma 3, there exist $j^{(\ell)} \in N$ and $R^{(\ell)}_{j^{(\ell)}} \in \mathcal{D}_{j^{(\ell)}}$ satisfying (lem3-1), (lem3-2) and (lem3-3). Then, by (lem2-1) or (lem3-1), {there is $t \in T_{j^{(\ell)}}$ s.t. $|\text{SUC}(R^{(\ell-1)t}_{j^{(\ell)}}, \emptyset^t)| \geq 2\}$ or $R^{(\ell-1)}_{j^{(\ell)}} \notin \mathcal{P}(X_{j^{(\ell)}})$. Moreover, by (lem2-2) or (lem3-2), $B(R^{(\ell-1)}) \geq B(R^{(\ell)})$. Therefore, if $R^{(\ell)}$ satisfies the (ζ,φ)-reverse property, then $(j^{(1)}, R^{(1)}_{j^{(1)}}), \cdots, (j^{(\ell)}, R^{(\ell)}_{j^{(\ell)}})$ is the desired sequence with its length ℓ. If $R^{(\ell)}$ does not satisfy the (ζ,φ)-reverse property, then go to the next step. Note that by (lem2-3) or (lem3-3), $\zeta(R^{(\ell)})$ dom$(R^{(\ell)})$ $\varphi(R^{(\ell)})$.

We claim that the above procedure stops in a finite number of steps, i.e., there exists $L \geq 1$ such that $R^{(L)}$ satisfies the (ζ,φ)-reverse property. Since in each step ℓ, by (lem2-2) or (lem3-2),

$$[B(R^{(\ell-1)}) \geq B(R^{(\ell)}) \text{ and } I(R^{(\ell-1)}) \geq I(R^{(\ell)})] \text{ and } [B(R^{(\ell-1)}) > B(R^{(\ell)}) \text{ or } I(R^{(\ell-1)}) > I(R^{(\ell)})].$$

Since $B(R^{(0)})$ and $I(R^{(0)})$ are non-negative integers, the procedure cannot have infinite steps. □

[39] Therefore, the induction argument bites.

Proof of Theorem 1

Suppose to the contrary that a *strategy-proof* rule ζ dominates φ. Let $R^{(1,0)} \in \mathcal{D}$ be such that $\zeta(R^{(1,0)})$ dom$(R^{(1,0)})$ $\varphi(R^{(1,0)})$. Let $b^{(1,0)} := \zeta(R^{(1,0)})$ and $a^{(1,0)} := \varphi(R^{(1,0)})$. Let $N^{(0)} := |N|$ and $B^{(1)} := B(R^{(1,0)})$. We will inductively construct two infinite sequences of non-negative integers $\{N^{(k)}\}_{k=0}^{\infty}$ and $\{B^{(k)}\}_{k=1}^{\infty}$ that cause a contradiction.

Induction step 1. If $R^{(1,0)}$ satisfies the (ζ, φ)-reverse property, then let $L_1 := 0$. If $R^{(1,0)}$ does not, then by Lemma 4, there exists a finite sequence of agent-preference pairs $\{(j^{(1,\ell)}, R^{(1,\ell)}_{j^{(1,\ell)}})\}_{\ell=1}^{L_1}$ satisfying the following (1-i), (1-ii) and (1-iii), where for each $\ell \in \{1, \cdots, L_1\}$, $R^{(1,\ell)} := (R^{(1,\ell)}_{j^{(1,\ell)}}; R^{(1,\ell-1)}_{-j^{(1,\ell)}})$.

(1-i) $\forall \ell = 1, \cdots, L_1$, $\big[\{\exists t \in T_{j^{(1,\ell)}} \text{ s.t. } |\text{SUC}(R^{(1,\ell-1)t}_{j^{(1,\ell)}}, \emptyset^t)| \geq 2\} \text{ or } R^{(1,\ell-1)}_{j^{(1,\ell)}} \notin \mathcal{P}(X_{j^{(1,\ell)}})\big]$,

(1-ii) $B(R^{(1,0)}) \geq B(R^{(1,1)}) \geq \cdots \geq B(R^{(1,L_1)})$, and

(1-iii) $R^{(1,L_1)}$ satisfies the (ζ, φ)-reverse property.

For each $\ell \in \{1, \cdots, L_1\}$, let $b^{(1,\ell)} := \zeta(R^{(1,\ell)})$ and $a^{(1,\ell)} := \varphi(R^{(1,\ell)})$.

Since $R^{(1,L_1)}$ satisfies the (ζ, φ)-reverse property,

$$\exists i^{(1)} \in N \text{ s.t. } \big[b^{(1,L_1)}_{i^{(1)}} \; P^{(1,L_1)}_{i^{(1)}} \; a^{(1,L_1)}_{i^{(1)}} \text{ and } \{\exists t \in T_{i^{(1)}} \text{ s.t. } a^{(1,L_1)t}_{i^{(1)}} \; P^{(1,L_1)t}_{i^{(1)}} \; b^{(1,L_1)t}_{i^{(1)}}\}\big]. \qquad \text{A}(1)$$

Let $N^{(1)} := |N \backslash \{i^{(1)}\}|$. Now, we change the agent $i^{(1)}$'s preference. Let $(\tilde{R}^{(2,0)t}_{i^{(1)}})_{t \in T_{i^{(1)}}}$ be a list of type preferences satisfying the $(1^* - i)$ below. By applying Lemma 1 for $(\tilde{R}^{(2,0)t}_{i^{(1)}})_{t \in T_{i^{(1)}}}$ and $(\emptyset^t)_{t \in T_{i^{(1)}}}$, we obtain a preference $R^{(2,0)}_{i^{(1)}} \in \mathcal{P}_{add}(X_{i^{(1)}})$ with $(R^{(2,0)t}_{i^{(1)}})_{t \in T_{i^{(1)}}} = (\tilde{R}^{(2,0)t}_{i^{(1)}})_{t \in T_{i^{(1)}}}$ satisfying $(1^* - ii)$.

(1*-i) $\forall t \in T_{i^{(1)}}, \forall x^t \in X^t \backslash \{b^{(1,L_1)t}_{i^{(1)}}, \emptyset^t\}, b^{(1,L_1)t}_{i^{(1)}} \; \tilde{R}^{(2,0)t}_{i^{(1)}} \; \emptyset^t \; \tilde{P}^{(2,0)t}_{i^{(1)}} \; x^t$ and

(1*-ii) $\forall x_{i^{(1)}} \in X_{i^{(1)}}, \big[\{\exists t \in T_{i^{(1)}} \text{ s.t. } x^t_{i^{(1)}} \in \text{SLC}(R^{(2,0)t}_{i^{(1)}}, \emptyset^t)\} \Rightarrow (\emptyset^t)_{t \in T_{i^{(1)}}} \; P^{(2,0)}_{i^{(1)}} \; x_{i^{(1)}}\big]$.

Let $R^{(2,0)} := (R^{(2,0)}_{i^{(1)}}; R^{(1,L_1)}_{-i^{(1)}})$, $b^{(2,0)} := \zeta(R^{(2,0)})$, $a^{(2,0)} := \varphi(R^{(2,0)})$ and $B^{(2)} := B(R^{(2,0)})$.

Claim 1.1. $b^{(2,0)}$ dom$(R^{(2,0)})$ $a^{(2,0)}$.

We show $b^{(2,0)}_{i^{(1)}} \; P^{(2,0)}_{i^{(1)}} \; a^{(2,0)}_{i^{(1)}}$. Suppose to the contrary that $b^{(2,0)}_{i^{(1)}} = a^{(2,0)}_{i^{(1)}}$.[40] Since ζ is *strategy-proof*, $b^{(2,0)}_{i^{(1)}} = \zeta_{i^{(1)}}(R^{(2,0)}) \; R^{(2,0)}_{i^{(1)}} \; \zeta_{i^{(1)}}(R^{(1,L_1)}) = b^{(1,L_1)}_{i^{(1)}}$. Note that by (1*-i) and the separability of $R^{(2,0)}_{i^{(1)}}$, $b^{(1,L_1)}_{i^{(1)}}$ is a best bundle at $R^{(2,0)}_{i^{(1)}}$. Thus, since $R^{(2,0)}_{i^{(1)}}$ is strict, $b^{(2,0)}_{i^{(1)}} = b^{(1,L_1)}_{i^{(1)}}$. Thus, $a^{(2,0)}_{i^{(1)}} = b^{(1,L_1)}_{i^{(1)}}$. Hence, by (A(1)), $\varphi_{i^{(1)}}(R^{(2,0)}) = a^{(2,0)}_{i^{(1)}} = b^{(1,L_1)}_{i^{(1)}} \; P^{(1,L_1)}_{i^{(1)}} \; a^{(1,L_1)}_{i^{(1)}} = \varphi_{i^{(1)}}(R^{(1,L_1)})$, a violation of *strategy-proofness* of φ. Therefore, $b^{(2,0)}_{i^{(1)}} \; P^{(2,0)}_{i^{(1)}} \; a^{(2,0)}_{i^{(1)}}$. Since ζ dom φ, this completes the proof of Claim 1.1.

The following claim trivially holds since $N^{(0)} > N^{(1)}$.

Claim 1.2. $N^{(0)} = N^{(1)} \Rightarrow B^{(1)} > B^{(2)}$.

Now, let $k \geq 2$.

[40] Note that $R^{(2,0)}_{i^{(1)}}$ is strict.

Induction hypothesis. Suppose that the series of statements below are true for each $k' < k$. If $\{(j^{(k',\ell)}, R^{(k',\ell)}_{j^{(k',\ell)}})\}_{\ell=1}^{L_{k'}}$ is not defined, $L_{k'} = 0$. If $\{(j^{(k',\ell)}, R^{(k',\ell)}_{j^{(k',\ell)}})\}_{\ell=1}^{L_{k'}}$ is defined, then it satisfies the following (k'-i), (k'-ii) and (k'-iii), where for each $\ell \in \{1, \cdots, L_{k'}\}$, $R^{(k',\ell)} := (R^{(k',\ell)}_{j^{(k',\ell)}}; R^{(k',\ell-1)}_{-j^{(k',\ell)}})$.

(k'-i) $\forall \ell = 1, \cdots, L_{k'}$, $\big[\{\exists t \in T_{j^{(k',\ell)}}$ s.t. $|\text{SUC}(R^{(k',\ell-1)t}_{j^{(k',\ell)}}, \emptyset^t)| \geq 2\}$ or $R^{(k',\ell-1)}_{j^{(k',\ell)}} \notin \mathcal{P}(X_{j^{(k',\ell)}})\big]$,

(k'-ii) $B(R^{(k',0)}) \geq B(R^{(k',1)}) \geq \cdots \geq B(R^{(k',L_{k'})})$, and

(k'-iii) $R^{(k',L_{k'})}$ satisfies the (ζ, φ)-reverse property.

For each $\ell = 1, \cdots, L_{k'}$, let $b^{(k',\ell)} := \zeta(R^{(k',\ell)})$ and $a^{(k',\ell)} := \varphi(R^{(k',\ell)})$. For agent $i^{(k')} \in N$,

$$\big[b^{(k',L_{k'})}_{i^{(k')}} \; P^{(k',L_{k'})}_{i^{(k')}} \; a^{(k',L_{k'})}_{i^{(k')}} \text{ and } \{\exists t \in T_{i^{(k')}} \text{ s.t. } a^{(k',L_{k'})t}_{i^{(k')}} \; P^{(k',L_{k'})t}_{i^{(k')}} \; b^{(k',L_{k'})t}_{i^{(k')}}\}\big]. \qquad \text{A(k')}$$

Let $N^{(k')} := |N \backslash \{i^{(1)}, \cdots, i^{(k')}\}|$. For agent $i^{(k')}$'s new preference $R^{(k'+1,0)}_{i^{(k')}} \in \mathcal{P}_{add}(X_{i^{(k')}})$,

(k'*-i) $\forall t \in T_{i^{(k')}}, \forall x^t \in X^t \backslash \{b^{(k',L_{k'})t}_{i^{(k')}}, \emptyset^t\}, b^{(k',L_{k'})t}_{i^{(k')}} \; R^{(k'+1,0)t}_{i^{(k')}} \; \emptyset^t \; P^{(k'+1,0)t}_{i^{(k')}} \; x^t$, and

(k'*-ii) $\forall x_{i^{(k')}} \in X_{i^{(k')}}, \big[\{\exists t \in T_{i^{(k')}}$ s.t. $x^t_{i^{(k')}} \in \text{SLC}(R^{(k'+1,0)t}_{i^{(k')}}, \emptyset^t)\} \Rightarrow (\emptyset^t)_{t \in T_{i^{(k')}}} \; P^{(k'+1,0)}_{i^{(k')}} \; x_{i^{(k')}}\big]$.

Letting $R^{(k'+1,0)} := (R^{(k'+1,0)}_{i^{(k')}}; R^{(k',L_{k'})}_{-i^{(k')}})$, $b^{(k'+1,0)} := \zeta(R^{(k'+1,0)})$, $a^{(k'+1,0)} := \varphi(R^{(k'+1,0)})$ and $B^{(k'+1)} := B(R^{(k'+1,0)})$.
Claim k'.1. $b^{(k'+1,0)} \; \text{dom}(R^{(k'+1,0)}) \; a^{(k'+1,0)}$, and
Claim k'.2. $N^{(k'-1)} = N^{(k')} \Rightarrow B^{(k')} > B^{(k'+1)}$.

Induction step *k*. If $R^{(k,0)}$ satisfies the (ζ, φ)-reverse property, then let $L_k := 0$. If $R^{(k,0)}$ does not, then by Lemma 4, there exists a finite sequence of agent-preference pairs $\{(j^{(k,\ell)}, R^{(k,\ell)}_{j^{(k,\ell)}})\}_{\ell=1}^{L_k}$ satisfying the following (k-i), (k-ii) and (k-iii), where for each $\ell = 1, \cdots, L_k$, $R^{(k,\ell)} := (R^{(k,\ell)}_{j^{(k,\ell)}}; R^{(k,\ell-1)}_{-j^{(k,\ell)}})$.

(k-i) $\forall \ell = 1, \cdots, L_k$, $\big[\{\exists t \in T_{j^{(k,\ell)}}$ s.t. $|\text{SUC}(R^{(k,\ell-1)t}_{j^{(k,\ell)}}, \emptyset^t)| \geq 2\}$ or $R^{(k,\ell-1)}_{j^{(k,\ell)}} \notin \mathcal{P}(X_{j^{(k,\ell)}})\big]$,

(k-ii) $B(R^{(k,0)}) \geq B(R^{(k,1)}) \geq \cdots \geq B(R^{(k,L_k)})$, and

(k-iii) $R^{(k,L_k)}$ satisfies the (ζ, φ)-reverse property.

For each $\ell = 1, \cdots, L_k$, let $b^{(k,\ell)} := \zeta(R^{(k,\ell)})$ and $a^{(k,\ell)} := \varphi(R^{(k,\ell)})$. Since $R^{(k,L_k)}$ satisfies the (ζ, φ)-reverse property,

$$\exists i^{(k)} \in N \text{ s.t. } \big[b^{(k,L_k)}_{i^{(k)}} \; P^{(k,L_k)}_{i^{(k)}} \; a^{(k,L_k)}_{i^{(k)}} \text{ and } \{\exists t \in T_{i^{(k)}} \text{ s.t. } a^{(k,L_k)t}_{i^{(k)}} \; P^{(k,L_k)t}_{i^{(k)}} \; b^{(k,L_k)t}_{i^{(k)}}\}\big]. \qquad \text{A(k)}$$

Let $N^{(k)} := |N \backslash \{i^{(1)}, \cdots, i^{(k)}\}|$. Now, we change the agent $i^{(k)}$'s preference. Let $(\tilde{R}^{(k+1,0)t}_{i^{(k)}})_{t \in T_{i^{(k)}}}$ be a list of type preferences satisfying the $(k^* - i)$ below. By applying Lemma 1 for $(\tilde{R}^{(k+1,0)t}_{i^{(k)}})_{t \in T_{i^{(k)}}}$ and $(\emptyset^t)_{t \in T_{i^{(k)}}}$, we obtain a preference $R^{(k+1,0)}_{i^{(k)}} \in \mathcal{P}_{add}(X_{i^{(k)}})$ with $(R^{(k+1,0)t}_{i^{(k)}})_{t \in T_{i^{(k)}}} = (\tilde{R}^{(k+1,0)t}_{i^{(k)}})_{t \in T_{i^{(k)}}}$ satisfying $(k^* - ii)$.

(k*-i) $\forall t \in T_{i^{(k)}}, \forall x^t \in X^t \backslash \{b_{i^{(k)}}^{(k,L_k)t}, \emptyset^t\}, b_{i^{(k)}}^{(k,L_k)t}\ \tilde{R}_{i^{(k)}}^{(k+1,0)t}\ \emptyset^t\ \tilde{P}_{i^{(k)}}^{(k+1,0)t}\ x^t$ and

(k*-ii) $\forall x_{i^{(k)}} \in X_{i^{(k)}}, \big[\{\exists t \in T_{i^{(k)}} \text{ s.t. } x_{i^{(k)}}^t \in \text{SLC}(R_{i^{(k)}}^{(k+1,0)t}, \emptyset^t)\} \Rightarrow (\emptyset^t)_{t\in T_{i^{(k)}}}\ P_{i^{(k)}}^{(k+1,0)}\ x_{i^{(k)}}\big]$.

Let $R^{(k+1,0)} := (R_{i^{(k)}}^{(k+1,0)}; R_{-i^{(k)}}^{(k,L_k)})$, $b^{(k+1,0)} := \zeta(R^{(k+1,0)})$, $a^{(k+1,0)} := \varphi(R^{(k+1,0)})$ and $B^{(k+1)} := B(R^{(k+1,0)})$.

<u>Claim k.1.</u> $b^{(k+1,0)}\ \text{dom}(R^{(k+1,0)})\ a^{(k+1,0)}$.

We show that $b_{i^{(k)}}^{(k+1,0)}\ P_{i^{(k)}}^{(k+1,0)}\ a_{i^{(k)}}^{(k+1,0)}$. Suppose to the contrary that $b_{i^{(k)}}^{(k+1,0)} = a_{i^{(k)}}^{(k+1,0)}$.[41] Since ζ is *strategy-proof*, $b_{i^{(k)}}^{(k+1,0)} = \zeta_{i^{(k)}}(R^{(k+1,0)})\ R_{i^{(k)}}^{(k+1,0)}\ \zeta_{i^{(k)}}(R^{(k,L_k)}) = b_{i^{(k)}}^{(k,L_k)}$. Note that by (k*-i) and the separability of $R_{i^{(k)}}^{(k+1,0)}$, $b_{i^{(k)}}^{(k,L_k)}$ is a best bundle at $R_{i^{(k)}}^{(k+1,0)}$. Since $R_{i^{(k)}}^{(k+1,0)}$ is strict, $b_{i^{(k)}}^{(k+1,0)} = b_{i^{(k)}}^{(k,L_k)}$. Thus, $a_{i^{(k)}}^{(k+1,0)} = b_{i^{(k)}}^{(k,L_k)}$. Hence, by (A(k)), $\varphi_{i^{(k)}}(R^{(k+1,0)}) = a_{i^{(k)}}^{(k+1,0)} = b_{i^{(k)}}^{(k,L_k)}\ P_{i^{(k)}}^{(k,L_k)}\ a_{i^{(k)}}^{(k,L_k)} = \varphi_{i^{(k)}}(R^{(k,L_k)})$, a violation of *strategy-proofness* of φ. Therefore, $b_{i^{(k)}}^{(k+1,0)}\ P_{i^{(k)}}^{(k+1,0)}\ a_{i^{(k)}}^{(k+1,0)}$. Since ζ dom φ, this completes the proof of Claim k.1.

<u>Claim k.2.</u> $N^{(k-1)} = N^{(k)} \Rightarrow B^{(k)} > B^{(k+1)}$.

Suppose that $N^{(k-1)} = N^{(k)}$, i.e., $i^{(k)} \in \{i^{(1)}, \cdots, i^{(k-1)}\}$. Let k' be the largest integer with $k' < k$ such that $i^{(k')} = i^{(k)}$. First, we claim that the preference replacement from $R_{i^{(k)}}^{(k,L_k)}$ to $R_{i^{(k)}}^{(k+1,0)}$ is the first opportunity to change $i^{(k)}(= i^{(k')})$'s preference after we have chosen $R_{i^{(k')}}^{(k'+1,0)}$. Formally,

<u>Claim k.2.1.</u> $R_{i^{(k)}}^{(k,L_k)} = R_{i^{(k')}}^{(k'+1,0)}$.

Suppose $R_{i^{(k)}}^{(k,L_k)} \neq R_{i^{(k')}}^{(k'+1,0)}$. By definition of k',

$$\exists k'' \in \{k'+1, \cdots, k\}, \exists \ell \in \{1, \cdots, L_{k''}\} \text{ s.t. } i^{(k')} = j^{(k'',\ell)}.^{42}$$

Let (k'', ℓ) be the first index in which $i^{(k')} = j^{(k'',\ell)}$.[43] However, by the condition (k"-i),

$$\{\exists t \in T_{j^{(k'',\ell)}} \text{ s.t. } |\text{SUC}(R_{j^{(k'',\ell)}}^{(k'',\ell-1)t}, \emptyset^t)| \geq 2\} \text{ or } R_{j^{(k'',\ell)}}^{(k'',\ell-1)} \notin \mathcal{P}(X_{j^{(k'',\ell)}})$$

while $R_{i^{(k')}}^{(k'+1,0)}(= R_{j^{(k'',\ell)}}^{(k'',\ell-1)})$ belongs to $\mathcal{P}(X_{i^{(k')}})$, and satisfies the condition (k'*-i) which asserts that for each $t \in T_{i^{(k')}}, |\text{SUC}(R_{i^{(k')}}^{(k'+1,0)t}, \emptyset^t)| \leq 1$, a contradiction. This completes the proof of Claim k.2.1.

The next claim asserts that for each $t \in T_{i^{(k)}}$, the type-t obeject at $b_{i^{(k)}}^{(k,L_k)}$ is at least as good as the type-t null object for agent $i^{(k)}$ according to $R_{i^{(k)}}^{(k,L_k)}$.

[41] Note that $R_{i^{(k)}}^{(k+1,0)}$ is strict.

[42] In words, the preference replacement is accompanied by the application of Lemma 4 in step k'' after k'.

[43] Namely, k'' is the smallest first coordinate among the pairs satisfying the condition, and ℓ is the smallest second coordinate among the pairs in which the first coordinate is k'' satisfying the condition.

Claim k.2.2. $\forall t \in T_{i^{(k)}}, b^{(k,L_k)t}_{i^{(k)}}\ R^{(k,L_k)t}_{i^{(k)}}\ \emptyset^t$.

Suppose to the contrary that there exists $t \in T_{i^{(k)}}$ such that $\emptyset^t\ P^{(k,L_k)t}_{i^{(k)}}\ b^{(k,L_k)t}_{i^{(k)}}$. By Claim k.2.1., $R^{(k,L_k)}_{i^{(k)}} = R^{(k'+1,0)}_{i^{(k')}}$. Thus, by the condition (k′*-ii), $(\emptyset^t)_{t\in T_{i^{(k)}}}\ P^{(k,L_k)}_{i^{(k)}}\ b^{(k,L_k)}_{i^{(k)}} = \zeta_{i^{(k)}}(R^{(k,L_k)})$. Thus, ζ is not *individually rational*. However, since ζ dom φ, ζ is *individually rational*, which is a contradiction. This completes the proof of Claim k.2.2.

Now, we turn back to the proof of Claim k.2. By the condition A(k) and Claim k.2.2., there is $t \in T_{i^{(k)}}$ such that $a^{(k,L_k)t}_{i^{(k)}}\ P^{(k,L_k)t}_{i^{(k)}}\ b^{(k,L_k)t}_{i^{(k)}}\ R^{(k,L_k)t}_{i^{(k)}}\ \emptyset^t$. Thus, $B(R^{(k,L_k)t}_{i^{(k)}}) > B(R^{(k+1,0)t}_{i^{(k)}})$. Note that by (k*-1) and Claim k.2.2., for each $t' \in T_{i^{(k)}}\setminus\{t\}$, $B(R^{(k,L_k)t'}_{i^{(k)}}) \geq B(R^{(k+1,0)t'}_{i^{(k)}})$. Summing up, we have $B(R^{(k,L_k)}_{i^{(k)}}) > B(R^{(k+1,0)}_{i^{(k)}})$. Thus, $B(R^{(k,L_k)}) > B(R^{(k+1,0)})$ since $i^{(k)}$'s preference is the only difference between $R^{(k,L_k)}$ and $R^{(k+1,0)}$. By the condition (k-ii), $B(R^{(k,0)}) \geq \cdots \geq B(R^{(k,L_k)})$. Therefore, $B^{(k)} = B(R^{(k,0)}) > B(R^{(k+1,0)}) = B^{(k+1)}$. This completes the proof of Claim k.2.

We have inductively defined two sequences $\{N^{(k)}\}_{k=0}^{\infty}$ and $\{B^{(k)}\}_{k=1}^{\infty}$ of non-negative integers. Obviously, $\{N^{(k)}\}_{k=0}^{\infty}$ is weakly decreasing, i.e., $N^{(0)} > N^{(1)} \geq \cdots N^{(k)} \geq \cdots$. We show that

$$\forall k \in \mathbb{N}, \exists k' > k \text{ s.t. } N^{(k)} > N^{(k')}. \tag{3}$$

Let $k \in \mathbb{N}$ be arbitrary. Let $K := 1 + B^{(k+1)}$ and $k' := k + K$. We prove by contradiction that $N^{(k)} > N^{(k')}$. If $N^{(k)} = N^{(k+1)} = \cdots = N^{(k+K)}$, then by Claims k.2 to (k+K-1).2., we can conclude that $B^{(k+1)} > B^{(k+2)} > \cdots > B^{(k+K+1)}$. This implies $0 > B^{(k+K+1)}$, a contradiction. Thus, (3) holds. However, $N^{(0)}$ is a finite non-negative number, a contradiction. This completes the proof of Theorem 1. □

B.1 Proof of Corollary 3

Next we prove Corollary 3. We begin with three lemmas.

Lemma 5. *Suppose that φ is independent, and for each $t \in T$, $\Phi^t : \mathcal{P}(X^t)^{N^t} \to \mathcal{A}^t$ is strategy-proof and Pareto efficient. Suppose also that a strategy-proof rule ζ dominates φ. Then, for each $R \in \mathcal{D}$, if $\zeta(R)$ dom $\varphi(R)$, then $\zeta(R)$ does not coordinate-wise weakly dominate $\varphi(R)$ at R.*

Proof. Suppose to the contrary that $\zeta(R)$ cw-dom(R) $\varphi(R)$. Since $\zeta(R)$ dom(R) $\varphi(R)$, there exists $i \in N$ such that $\zeta_i(R)\ P_i\ \varphi_i(R)$. Since R_i is separable, there exists $t \in T_i$ such that $\zeta^t_i(R)\ P^t_i\ \varphi^t_i(R) = \Phi^t_i(R^t)$. Since $\zeta(R)$ cw-dom(R) $\varphi(R)$, for each $j \in N^t\setminus\{i\}$, $\zeta^t_j(R)\ R^t_j\ \varphi^t_j(R) = \Phi^t_j(R^t)$. This violates that $\Phi^t(R^t)$ is *Pareto efficient* at R^t. □

Lemma 6. *Suppose that φ is independent and for each $t \in T$, $\Phi^t : \mathcal{P}(X^t)^{N^t} \to \mathcal{A}^t$ is strategy-proof and Pareto efficient. Suppose also that a strategy-proof rule ζ dominates φ. Let $R \in \mathcal{D}$ be such that*

$\zeta(R)$ $dom(R)$ $\varphi(R)$. *If* R *does not satisfy the* (ζ,φ)*-reverse property, then there exist* $i \in N$ *and* $R_i' \in \mathcal{D}_i$ *such that*

(lem6-1) $R_i \notin \mathcal{P}(X_i)$,

(lem6-2) $I(R_i'; R_{-i}) < I(R)$ *and* $(R_i'^t)_{t\in T_i} = (R_i^t)_{t\in T_i}$, *and*

(lem6-3) $\zeta(R_i'; R_{-i})$ $dom(R_i'; R_{-i})$ $\varphi(R_i'; R_{-i})$.

Proof. Same as the proof of Lemma 3. □

Lemma 7. *Suppose that a rule* φ *is independent and for each* $t \in T$, $\Phi^t : \mathcal{P}(X^t)^{N^t} \to \mathcal{A}^t$ *is strategy-proof and Pareto efficient. Suppose also that a strategy-proof rule* ζ *dominates* φ*. Let* $R^{(0)} \in \mathcal{D}$ *be such that* $\zeta(R^{(0)})$ $dom(R^0)$ $\varphi(R^{(0)})$*. If* $R^{(0)}$ *does not satisfy the* (ζ,φ)*-reverse property, then there exists a finite sequence of agent-preference pairs* $\{(j^{(\ell)}, R^{(\ell)}_{j^{(\ell)}})\}_{\ell=1}^L$ *satisfying the following conditions (i), (ii) and (iii). For each* $\ell = 1,\cdots,L$*, let* $R^{(\ell)} := (R^{(\ell)}_{j^{(\ell)}}; R^{(\ell-1)}_{-j^{(\ell)}})$.

(i) $\forall \ell = 1,\cdots,L$, $R^{(\ell-1)}_{j^{(\ell)}} \notin \mathcal{P}(X_{j^{(\ell)}})$,

(ii) $\forall \ell = 1,\cdots,L$, $(R^{(\ell)t}_{j^{(\ell)}})_{t\in T_{j^{(\ell)}}} = (R^{(\ell-1)t}_{j^{(\ell)}})_{t\in T_{j^{(\ell)}}}$ *and*

(iii) $R^{(L)}$ *satisfies the* (ζ,φ)*-reverse property.*

Proof. Same as the proof of Lemma 4. □

We introduce notations: For each $i \in N$, each $t \in T_i$, each $R_i^t \in \mathcal{P}(X^t)$, each $R_i \in \mathcal{D}_i$ and each $R \in \mathcal{D}$, let $B^\omega(R_i^t) := |\mathrm{SUC}(R_i^t, \omega_i^t)|$, $B^\omega(R_i) := \sum_{t\in T_i} B^\omega(R_i^t)$, and $B^\omega(R) := \sum_{i\in N} B^\omega(R_i)$. The operator B^ω assigns the number of object(s) which are preferred to the endowed object(s). Now we are ready to prove Corollary 3.

Proof of Corollary 3

Replacing B in the proof of Theorem 1 with B^ω, the same proof as that for Theorem 1 works, where for each $i \in N$, ω_i plays the role of $(\emptyset^t)_{t\in T_i}$ in the proof of Theorem 1. □

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