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Abstract

On the (im)possibility of improving upon the student-proposing deferred acceptance mechanism

by Onur Kesten and Morimitsu Kurino*

This paper studies a general school choice problem with or without outside options. The Gale-Shapley student-proposing deferred acceptance mechanism (DA) has played a central role not only in theory but also in important practical applications. We show that in problems where some students cannot credibly submit a single school as the only acceptable option, it is possible to improve upon DA without sacrificing strategy-proofness. On the other hand, in unrestricted problems where no outside options necessarily exist, it is not possible to improve upon DA via a strategy-proof mechanism.

Keywords: Student-proposing deferred acceptance mechanism, strategy-proofness, Pareto dominance, outside options

JEL classification: C78, D78, I21

Zusammenfassung

Es wird ein allgemeines school-choice-Problem untersucht mit und ohne „outside options". Der „Gale-Shapley student-proposing deferred acceptance"-Mechanismus (DA) spielt eine zentrale Rolle nicht nur in der Theorie sondern auch in wichtigen praktischen Anwendungen. Wir zeigen, dass in Situationen, in denen einige Schüler nicht glaubwürdig nur eine einzige Schule als akzeptable Option angeben können, der DA verbessert werden kann, ohne dass die Nichtmanipulierbarkeit („strategy proofness") des Mechanismus dafür geopfert werden muss. Auf der anderen Seite gilt für Situationen, in denen nicht notwendigerweise "outside options" existieren, dass der DA nicht verbessert werden kann mit Hilfe eines nichtmanipulierbaren Mechanismus.

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1 Introduction

The Gale-Shapley student-proposing deferred acceptance mechanism (DA) has been a prominent allocation method both in theory and in practice for school choice. Thanks to economists' and officials' collaborative efforts, the New York City Department of Education as well as the Boston Public School system transitioned to new designs implementing DA for student assignment beginning in 2004 and 2006, respectively. Because of its theoretical appeal, DA has remained at the center of the recent debate concerning the trade-off among efficiency, stability, and strategy-proofness.

In a model where schools have strict priorities, Kesten (2010) showed that no efficient and strategy-proof mechanism dominates DA. In a two-sided matching model allowing for weak priorities, Abdulkadiroğlu, Pathak, and Roth (2009) (henceforth APR) showed that no strategy-proof mechanism dominates DA. APR claimed that the latter result is tighter than the former. We note that Kesten's model cannot be embedded into the APR framework. Nor is the proof technique offered by APR applicable in this setting. Critically, contrary to Kesten, APR assume that students are equipped with the strategy to rank-list an option of remaining "unassigned" and are able to submit preference lists that declare only a single school as acceptable. Nevertheless, singleton preference lists are risky and are generally used only by a small fraction of students (see Figure 1). Moreover, most if not all households typically do not have the means to afford outside options such as private education.

We study two natural departures from the APR framework to investigate the strategic role of outside options. First, we consider problems where some students cannot credibly submit singleton preference lists. Specifically, these are problems that involve at least two students who draw their reported top-choices from a set T of schools where $|T| \geq 2$ and T does not contain any outside options. A simple problem of this kind arises when there are at least two students who always report remaining unassigned as the last choice. We show that in such cases strategy-proof mechanisms dominating DA may exist (Theorem 1).[1]

Second, we consider unrestricted problems where no outside options necessarily exist. We show that in these cases DA cannot be dominated by a strategy-proof mechanism (Theorem 2). This result strengthens the impossibilities of Kesten (2010) and APR, and relies on a new proof that involves the identification of "overdemanded" schools that may be endogenously determined at a problem.

Our analysis suggests that manipulating a Pareto-superior mechanism to DA may be sensitive to a student's actual strategy set, and even in cases when profitable manipulation is possible, this

[1]For an assignment model that also assumes outside options, Erdil (2011) shows that no strategy-proof mechanism dominating a non-wasteful and strategy-proof mechanism exists. We note, however, that that result also no longer holds once outside options are ruled out. Consider, for example, a simple assignment setting with n agents and n objects. Clearly, any constant assignment mechanism is strategy-proof. However, such a mechanism is dominated by the corresponding core mechanism, which is also strategy-proof.

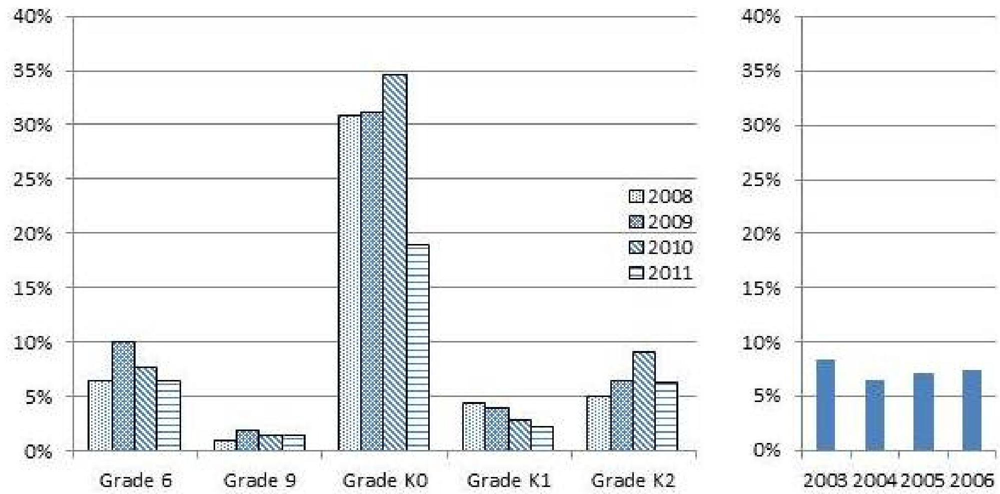

Figure 1: Percentage of students who listed only one school in Boston (left) and New York City (right). *Data*: The above plotted statistics for the Boston Public Schools and NYC are reported in Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006) and Abdulkadiroğlu, Pathak, and Roth (2009) respectively.

may require students to devise rather sophisticated strategies despite holding complete information about the environment.

2 The Model

We consider a general school choice model with or without the option of being unassigned. For convenience, we mostly follow the same notation as in APR. A school choice problem is a five-tuple $(I, S, (q_s)_{s\in S}, (P_i)_{i\in I}, (R_s)_{s\in S})$. I is a finite set of students and S is a finite set of schools. Each school $s \in S$ has q_s available seats, or **capacity**. We assume throughout the paper that the total number of seats is no less than the number of students, i.e., $|I| \leq \sum_{s\in S} q_s$. If q_s is large enough, say $q_s = |I|$, then the school s may represent the option of remaining unassigned (or, an outside option such as a private school). We call such a school the **null school**, denoted by $\emptyset$. We do not necessarily assume the existence of the null school. Notice that when the null school is allowed, this model contains that of APR. Each student $i \in I$ has a strict preference relation P_i on S. Let R_i denote the at-least-as-good-as relation associated with P_i. We assume that each school s has a weak priority R_s on I that is a complete and transitive binary relation on I. We say that a priority R_s is strict if it is an antisymmetric weak priority. Let P_s represent the asymmetric part of R_s. For any $I' \subseteq I$, let $P_{I'} = (P_i)_{i\in I'}$ and $P_{-i} = (P_j)_{j\in I\setminus\{i\}}$. We define $R_{S'}$ and R_{-s} similarly. We fix $I, S, (q_s)_{s\in S}$ throughout the paper.

A **matching** is a correspondence $\mu : I \cup S \to S \cup I$ such that each student is assigned only one school and each school is assigned students up to its capacity, i.e., for all $i \in I$ and $s \in S$, $\mu(i) \in S$, $\mu(s) \in I$, $|\mu(i)| = 1$, $|\mu(s)| \leq q_s$, and $i \in \mu(s) \Leftrightarrow s \in \mu(i)$. Since $\mu(i)$ is a singleton, we denote $\mu(i) = s$ instead of $\mu(i) = \{s\}$. A matching is **non-wasteful** at P_I if for all $i \in I$ and all $s \in S$, $s\ P_i\ \mu(i)$ implies $|\mu(s)| = q_s$. A matching μ **dominates** matching ν at P_I if for all $i \in I$, $\mu(i)\ R_i\ \nu(i)$, and for some $i \in I$, $\mu(i)\ P_i\ \nu(i)$. A matching is **Pareto efficient** at P_I if it is not dominated by any other matching at P_I. A pair $(i, s) \in I \times S$ **blocks** a matching μ at (P_I, R_S) if $s\ P_i\ \mu(i)$ and $\big[$either $|\mu(s)| < q_s$ or for some $j \in \mu(s)$, $i\ P_s\ j\ \big]$. A matching is **stable** at (P_I, R_S) if it is not blocked by any pair at (P_I, R_S). A stable matching is a **student-optimal stable matching** at (P_I, R_S) if it is not dominated at P_I by any other matching that is stable at (P_I, R_S).

A (direct) **mechanism** φ is a function that maps every (P_I, R_S) to a matching. Denote by $\varphi_i(P_I, R_S)$ the school that is matched to i by φ. Similarly, denote by $\varphi_s(P_I, R_S)$ the set of students that are matched to s by φ. A mechanism φ is **strategy-proof** if for all (P_I, R_S), $i \in I$, and P'_i, $\varphi_i(P_I, R_S)\ R_i\ \varphi_i(P'_i, P_{-i}, R_S)$. A mechanism φ **dominates** ψ if (i) for all P_I, R_S, and $i \in I$, $\varphi_i(P_I, R_S)\ R_i\ \psi_i(P_I, R_S)$, and (ii) for some P_I, R_S, and $i \in I$, $\varphi_i(P_I, R_S)\ P_i\ \psi_i(P_I, R_S)$. A mechanism φ is **Pareto efficient** (**non-wasteful**) if for all P_I and R_S, matching $\varphi(P_I, R_S)$ is Pareto efficient (non-wasteful) at P_I.

Abdulkadiroğlu and Sönmez (2003) advocated the use of Gale and Shapley's (1962) **student-proposing deferred acceptance (DA) algorithm** as a plausible assignment method in school choice. For a *strict* priority profile R_S, the algorithm works as follows:

Step 1: Each student applies to her favorite school. Each school s tentatively assigns its seats to its applicants following the priority order R_s. Any unassigned student is rejected.

In general,

Step k: Each student who was rejected at the previous step applies to her next favorite school. Each school s considers the students it has been holding together with its new applicants and tentatively assigns its seats following the priority order R_s. Any unassigned student is rejected.

The algorithm terminates when no student remains unassigned. At this point all current assignments are final.

A **tie-breaker** for school s is an injective function $\tau_s : I \to \mathbf{N}$ by associating R_s with a strict priority R^τ_s as follows: $i\ P^\tau_s\ j \Leftrightarrow [(i\ R_s\ j)$ or $(i\ I_s\ j$ and $\tau_s(i) < \tau_s(j))]$. A **tie-breaking rule** is a profile $\tau := (\tau_s)_{s \in S}$ of all schools' tie-breakers.

The **student-proposing deferred acceptance mechanism with a tie-breaking rule** τ, which is denoted by DA^τ, is the mechanism obtained by the student-proposing deferred acceptance algorithm acting on (P_I, R^τ_S), where R^τ_s is obtained from R_s by breaking ties using τ_s.

It is well known that if R_S is not strict, there might be multiple student-optimal stable matchings (Erdil and Ergin, 2008). But if it is strict, such a matching is unique and dominates any other stable matching (Gale and Shapley, 1962; Balinski and Sönmez, 1999). More precisely, when R_S is not strict and τ is a tie-breaking rule, $DA^\tau(P_I, R_S)$ is the unique student-optimal stable matching at (P_I, R_S^τ) and dominates at P_I any other matching that is stable at (P_I, R_S^τ). Furthermore, DA^τ is strategy-proof [2] (Dubins and Freedman, 1981; Roth, 1982) and non-wasteful.

3 The Main Results

We investigate whether or not there is a strategy-proof mechanism that dominates the student-proposing deferred acceptance mechanism with any tie-breaking rule. To this end, we introduce a key notion: A school s is **overdemanded at** $(P_I, R_S; \tau)$ if there is a student $i \in I$ such that $s \; P_i \; DA_i^\tau(P_I, R_S)$. A school s is **underdemanded at** $(P_I, R_S; \tau)$ if it is not overdemanded at $(P_I, R_S; \tau)$, i.e., for each student $i \in I$, $DA_i^\tau(P_I, R_S) \; R_i \; s$. Under DA an overdemanded school rejects at least one student, whereas an underdemanded school accepts all its applicants. Note that the null school is always underdemanded if it is allowed in the model.[3] Importantly, whether a school is overdemanded or not depends on the specific problem and the DA algorithm.

Lemma 1. *For all $(P_I, R_S; \tau)$, there is an underdemanded school at $(P_I, R_S; \tau)$.*

The proofs of all lemmas in this section are in the Appendix.

In our first departure from APR's specification of a school choice problem, we introduce a new type of students whose most preferred choices belong to a set T of "competitive" schools. We require that the size of T is at least two and these students never rank-list an option outside T as one of their top-$|T|$ choices. The former requirement implies that this student type cannot submit singleton preference lists, whereas the latter means that their strategic choices are limited to only those schools in the set T that cannot include the null school.[4] Our first result makes this description more precise, and shows that in these situations improving upon DA has no incentive cost.

Theorem 1. (Possibility) *Suppose that there exist a set J of at least two students and a set T of at least two schools such that the top-$|T|$ reported choices of each student in J belong to T and $\sum_{t \in T} q_t < |I|$.[5] Then, there is a strategy-proof mechanism that dominates the student-proposing deferred acceptance mechanism with any tie-breaking rule.*

[2] We note, however, that irrespective of the assumption on the existence of the null school, strategy-proofness of DA is lost when a quota is imposed on the length of students' preference lists. Therefore, consistent with the model of APR, we also assume that students face no such restrictions. See Haeringer and Klijn (2009) for an equlibrium analysis of DA when students' choices are constrained in this manner.

[3] Put differently, in our general model an underdemanded school is the analogue of the null school (or, the option of being unassigned) in the APR model. Therefore, here too a student need not rank-list any schools below an underdemanded school, as she would not need to rank-list any schools below the null school in the APR model.

[4] The set T can be interpreted as a group of schools that this type of students are seriously targeting such that they are not willing to take the risk of ranking a school from set T below a school they actually deem to be inferior.

[5] Put another way, the first part of the assumption suggests a sense of "correlation" among the preferences of students in J and the second part of the assumption (i.e., the inequality $\sum_{t \in T} q_t < |I|$) suggests a sense of "scarcity" for the collection of schools in T.

Proof. Let τ be a tie-breaking rule and $T := \{t_1, \dots, t_m\}$ such that $m \geq 2$ and $\sum_{t \in T} q_t < |I|$. Let two distinct students $i_1 \in J$ and $i_2 \in J$ report preferences that always list a school in T above any school not in T. Also, let $I_1, ..., I_{m+1}$ be a partition of I such that $|I_1| = q_{t_1}$,..., $|I_m| = q_{t_m}$, $i_1 \in I_1$, and $i_2 \in I_2$. Note that since $\sum_{t \in T} q_t < |I|$, I_{m+1} is nonempty. Let $\bar{R}_S$ be a priority profile such that the post-tie-breaking priority profile $\bar{R}^\tau_S$ is given by the following table on the right, where vertical dots represent arbitrary students. For example, the first column means that students in I_1 have higher priority for t_1 than those in I_{m+1}, who have higher priority than those in I_2, and the priorities of all remaining students are arbitrary. Similarly, for each school $t_j \in \{t_3, \dots, t_m\}$, students in I_j have the highest priority for it.

$P^{**}_{i \in I_1}$	$P^{*}_{i \in I_2}$	$P^{3}_{i \in I_3}$	$\cdots$	$P^{m}_{i \in I_m}$	$P^{*}_{i \in I_{m+1}}$
$\vdots$	$\vdots$	t_3	$\cdots$	t_m	$\vdots$
t_2	t_1	$\vdots$		$\vdots$	t_1
$\vdots$	$\vdots$				$\vdots$
t_1	t_2				t_2
$\vdots$	$\vdots$				$\vdots$

$\bar{R}^\tau_{t_1}$	$\bar{R}^\tau_{t_2}$	$\bar{R}^\tau_{t_3}$	$\bar{R}^\tau_{t_4}$	$\dots$	$\bar{R}^\tau_{t_m}$	$\bar{R}_{s \in S \setminus T}$
I_1	I_2	I_3	I_4	$\cdots$	I_m	$\vdots$
I_{m+1}	I_1	$\vdots$	$\vdots$		$\vdots$	
I_2	$\vdots$					
$\vdots$						

We consider the following sets of preferences (see the table above on the left): Let $\mathcal{P}^*$ be the set of preferences on S such that t_1 is preferred to t_2 and any school in T is preferred to each school not in T; let $\mathcal{P}^{**}$ be the set of preferences on S such that t_2 is preferred to t_1 and any school in T is preferred to each school not in T; for each $j = 3, \dots, m$, let $\mathcal{P}^j$ be the set of preferences on S such that t_j is the top choice; finally, let $\mathcal{P}$ be the set of all preferences on S. Note that for each student $i \in \{i_1, i_2\}$ and each P_i, because $i \in J$, we have $P_i \in \mathcal{P}^*$ or $P_i \in \mathcal{P}^{**}$. For each $I' \subseteq I$ and each set of preferences $\mathcal{P}'$, we denote by $\mathcal{P}'_{I'}$ the product set of $\mathcal{P}'$ over I'.

Then, it is straightforward to calculate that, for all $P_I \in (P^{**}_{I_1}, \mathcal{P}^*_{I_2}, \mathcal{P}^3_{I_3}, \dots, \mathcal{P}^m_{I_m}, \mathcal{P}^*_{I_{m+1}})$,

$$DA^\tau(P_I, \bar{R}_S) = \begin{pmatrix} I_1 & I_2 & \dots & I_m & I_{m+1} \\ t_1 & t_2 & \dots & t_m & S \setminus T \end{pmatrix}, \tag{1}$$

where all schools in T are overdemanded, and thus students i_1 and i_2 do not list any underdemanded schools above their DA assignments. We define a mechanism φ as follows: for all P_I and R_S, if $R_S \neq \bar{R}_S$, let $\varphi(P_I, R_S) = DA^\tau(P_I, R_S)$. If $R_S = \bar{R}_S$, $\varphi(P_I, R_S)|_{-\{i_1,i_2\}} = DA^\tau(P_I, R_S)|_{-\{i_1,i_2\}}$, and

$$\varphi(P_I, R_S)|_{\{i_1,i_2\}} = \begin{cases} \begin{pmatrix} i_1 & i_2 \\ t_2 & t_1 \end{pmatrix} & \text{if } P_I \in (P^{**}_{I_1}, \mathcal{P}^*_{I_2}, \mathcal{P}^3_{I_3}, \dots, \mathcal{P}^m_{I_m}, \mathcal{P}^*_{I_{m+1}}), \\ DA^\tau(P_I, R_S)|_{\{i_1,i_2\}} & \text{otherwise.} \end{cases}$$

Clearly, φ dominates DA^τ.[6] We show in the Online Appendix that φ is strategy-proof. □

Theorem 1 entails that once the ability of using the null school as a strategic option is limited for some students, the tension posed by strategy-proofness may be relaxed. As a realistic scenario, it is also plausible to imagine student types who cannot credibly use the null school as part of their strategies. The following corollary pertains to such cases where a school choice problem involves students who always report the null school as the least preferred outcome.[7] Then, we have the following corollary:

Corollary 1. *Suppose that there is the null school $\emptyset$, $|S \setminus \{\emptyset\}| \geq 2$, and $\sum_{s \in S\setminus\{\emptyset\}} q_s < |I|$. If the null school is the last reported choice of at least two students, there is a strategy-proof mechanism that dominates the student-proposing deferred acceptance mechanism with any tie-breaking rule.*

Now we turn to a more general model possibly without the null school, where a student can list any underdemanded school at any position of her preference list. In this case, one cannot improve upon DA without harming students' incentives.

Theorem 2. (Impossibility) *No strategy-proof mechanism dominates the student-proposing deferred acceptance mechanism with any tie-breaking rule.*

Corollary 2 (Kesten, 2010)**.** *When priorities are strict and the null school need not exist, no strategy-proof and efficient mechanism dominates the student-proposing deferred acceptance mechanism.*

Corollary 3 (Abdulkadiroğlu, Pathak, and Roth, 2009)**.** *When the null school exists, no strategy-proof mechanism dominates the student-proposing deferred acceptance mechanism with any tie-breaking rule.*

Remark 1. Note that Corollary 2 does not follow from Corollary 3 since APR's proof of Corollary 3 crucially depends on the existence of the null school.[8]

The following lemma says that if a matching under DA is dominated by another matching and some student i is assigned to different schools at the two matchings, then the assignment of student i under DA is overdemanded.

Lemma 2. *Suppose that a matching ν dominates $DA^\tau(P_I, R_S)$ at P_I for a given tie-breaking rule τ. If $\nu(i) \neq DA_i^\tau(P_I, R_S)$ for some $i \in I$, then school $DA_i^\tau(P_I, R_S)$ is overdemanded at $(P_I, R_S; \tau)$.*

[6] As shown in this construction, the DA outcome can be inefficient. Ergin (2002) identifies restrictions on priority structures to ensure the efficiency of DA.

[7] Indeed, in all years of Boston and NYC student assignments for which data exist, thousands of students chose to rank-list as many schools as they were allowed to.

[8] To illustrate the difference between the two setups through a realistic scenario, consider, for example, a poor neighborhood where the total capacity is sufficient to serve the student body within the neighborhood ($\sum_{s \in S\setminus\{\emptyset\}} q_s > |I|$) but students have no outside options. Notice that Corollary 3 is not applicable in this case. Moreover, Theorem 2 implies that the efficiency requirement in Corollary 2 can also be dropped.

We shall focus on a specific preference manipulation. To this end, we define some useful notions: Let $U(P_i, s) := \{s' \in S \mid s' \; R_i \; s\}$ be the **upper contour set of i at s**. Also, for any $S' \subseteq S$, define a strict preference relation $P_i|_{S'}$ on S' if for all $s, s' \in S'$, $s' \; P_i|_{S'} \; s \Leftrightarrow s' \; P_i \; s$. Given preference P_i of student i and schools s^*, s^u with $s^* \; P_i \; s^u$, **preference P'_i upgrades s^u above s^* in P_i** if P'_i ranks s^u right above s^* and the relative ranking of the other schools stays the same, i.e., (i) $s^* \; P_i \; s^u$ and $s^u \; P'_i \; s^*$, (ii) there is no $s \in S$ with $s^u \; P'_i \; s \; P'_i \; s^*$, and (iii) $P_i|_{S \setminus \{s^u, s^*\}} = P'_i|_{S \setminus \{s^u, s^*\}}$.

Lemma 3. *Suppose that student i is assigned school s^* that is overdemanded at $(P_I, R_S; \tau)$. Then, there is an underdemanded school s^u at $(P_I, R_S; \tau)$ such that $DA^\tau_i(P''_i, P_{-i}, R_S) = s^u$ and s^u is underdemanded at $(P''_i, P_{-i}, R_S; \tau)$, where preference P''_i upgrades s^u above s^* in P_i.*

We are now ready to prove Theorem 2.

Proof of Theorem 2. Fix a tie-breaking rule τ. Suppose that a strategy-proof mechanism φ dominates DA^τ. Then, there exist P_I, R_S, and $i \in I$ such that $\varphi_i(P_I, R_S) \; P_i \; DA^\tau_i(P_I, R_S)$. Let $s^* := DA^\tau_i(P_I, R_S)$. Since $\varphi_i(P_I, R_S) \neq s^*$, Lemma 2 implies that s^* is overdemanded at $(P_I, R_S; \tau)$. Thus, by Lemma 3, there is an underdemanded school s^u at $(P_I, R_S; \tau)$ such that school $DA^\tau_i(P''_i, P_{-i}, R_S) = s^u$ is underdemanded at $(P''_i, P_{-i}, R_S; \tau)$, where preference P''_i upgrades s^u above s^* in P_i. For simplicity, let $P'' := (P''_i, P_{-i})$.

We first show that $\varphi_i(P_I, R_S) = \varphi_i(P''_I, R_S)$. Note that $U(P_i, \varphi_i(P_I, R_S)) = U(P''_i, \varphi_i(P_I, R_S))$ and $P_i|_{U(P_i, \varphi_i(P_I, R_S))} = P''_i|_{U(P''_i, \varphi_i(P_I, R_S))}$, as P''_i upgrades s^u above s^* in P_i and $\varphi_i(P_I, R_S) \neq s^u, s^*$. By *strategy-proofness* of φ, $\varphi_i(P_I, R_S) \; R_i \; \varphi_i(P''_I, R_S)$. Also, $\varphi_i(P''_I, R_S) \; R''_i \; \varphi_i(P_I, R_S)$ and thus $\varphi_i(P''_I, R_S) \; R_i \; \varphi_i(P_I, R_S)$ as $P_i|_{U(P_i, \varphi_i(P_I, R_S))} = P''_i|_{U(P''_i, \varphi_i(P_I, R_S))}$. Thus, $\varphi_i(P_I, R_S) = \varphi_i(P''_I, R_S)$.

Now, we have $\varphi_i(P''_I, R_S) \neq DA^\tau_i(P''_I, R_S)$, for otherwise, since $\varphi_i(P_I, R_S) \; P_i \; DA^\tau_i(P_I, R_S)$ and $\varphi_i(P_I, R_S) = \varphi_i(P''_I, R_S)$, $DA^\tau_i(P''_I, R_S) \; P_i \; DA^\tau_i(P_I, R_S)$, which violates *strategy-proofness* of DA^τ.

Since φ dominates DA^τ, $\varphi(P''_I, R_S)$ dominates $DA^\tau(P''_I, R_S)$ at P''_I and $\varphi_i(P''_I, R_S) \neq DA^\tau_i(P''_I, R_S)$. Then, at (P''_I, R_S, τ), $DA^\tau_i(P''_I, R_S) \equiv s^u$, which is overdemanded by Lemma 2, whereas it is not by Lemma 3. A contradiction. □

4 Concluding Remarks

The Gale-Shapley deferred acceptance (DA) mechanism has been a focal assignment tool not only for theory but also for practical market design. Recent research has shown a surge of interest in exploring mechanisms that go beyond DA in terms of welfare (either ex ante or ex post). We have shown that whether such attempts come at the cost of strategy-proofness may be sensitive to the specifics of the environment. In circumstances when students cannot credibly use outside options as strategic choices or when it may be difficult to identify underdemanded schools, the scope of manipulation under alternative mechanisms may be diminished.

More broadly, our approach puts the three-way tension among efficiency, stability, and strategy-proofness into a new perspective by highligting the importance of the strategic role outside options may play. It remains an interesting future issue to search for new assignment mechanisms in light of this optimistic perspective.

A Appendix

A.1 Proof of Lemma 1

Let $(P_I, R_S; \tau)$ be given. Suppose on the contrary that all schools are overdemanded at $(P_I, R_S; \tau)$. Then, for each school $s \in S$, there is $j \in I$ such that $s \ P_j \ DA^{\tau}_j(P_I, R_S)$. Thus, by *non-wastefulness* of DA, $|DA^{\tau}_s(P_I, R_S)| = q_s$ for all $s \in S$. Note first that DA ends in at least two steps. Consider the last step $r \geq 2$ of DA at $(P_I, R_S; \tau)$ where some student k rejected at step $r-1$ and applies to some school t at step r. Since t is overdemanded, there is some student $k' \neq k$ such that $t \ P_{k'} \ DA^{\tau}_{k'}(P_I, R_S)$. Thus, k' must have applied to and been rejected by t at an earlier step than r. Thus, school t has kept q_t applicants at step $r-1$. Hence, school t has at least $|q_t|+1$ applications at step r, and thus rejects one of them. But this contradicts the assumption that r is the last step of DA. ■

A.2 Proof of Lemma 2

Suppose that matching ν dominates a non-wasteful matching μ at P_I. Let J be the set of students who prefer ν to μ, i.e., $J = \{i \in I \mid \nu(i) \ P_i \ \mu(i)\} = \{i \in I \mid \nu(i) \neq \mu(i)\}$. Define an **improvement cycle from μ to ν (at P_I)** as a finite list $(i^1, i^2, \ldots, i^n)$ of students, where $i^{n+1} \equiv i^1$ and $n \geq 2$, such that for each $k \in \{1, \ldots, n\}$, $\nu(i^k) = \mu(i^{k+1})$ and $\nu(i^k) \ P_{i^k} \ \mu(i^k)$.

Claim 1. Suppose that matching ν dominates a non-wasteful matching μ' at P_I. Then there is an improvement cycle from μ' to ν at P_I.

Proof. We define a list of students $(i^1, \ldots, i^n)$ to be **temporary list of size** $n \geq 2$ if for all $k \in \{1, \ldots, n\}$, $\nu(i^k) \ P_{i^k} \ \mu'(i^k)$ and for all $k \in \{1, \ldots, n-1\}$, $\nu(i^k) = \mu'(i^{k+1})$.

We first construct a temporary list of size 2. Since ν dominates μ' at P_I, there is $i^1 \in I$ such that $\nu(i^1) \ P_{i^1} \ \mu'(i^1)$. Let $s^2 := \nu(i^1)$ and $s^1 := \mu'(i^1)$. Note that $s^1 \neq s^2$. By *non-wastefulness* of μ', as $s^2 \ P_{i^1} \ \mu'(i^1)$, we have $|\mu'(s^2)| = q_{s^2}$. Then, there is $i^2 \in \mu'(s^2)$ such that $\nu(i^2) \neq \mu'(i^2)$ (otherwise, as $|\mu'(s^2)| = q_{s^2}$, we have $\mu'(s^2) = \nu(s^2)$ and thus $i^1 \in \nu(s^2) = \mu'(s^2)$. Thus, $\mu'(i^1) = s^2$ but $\mu'(i^1) \equiv s^1$, which contradicts $s^1 \neq s^2$.). Thus, as ν dominates μ' at P_I, $\nu(i^2) \ P_{i^2} \ \mu(i^2)$. Also, as $i^2 \in \mu'(s^2)$, $\nu(i^1) \equiv s^2 = \mu'(i^2)$. Thus, (i^1, i^2) is a temporary list of size 2. If $\nu(i^2) = \mu(i^1)$, the temporary list is an improvement cycle from μ' to ν. If not, we continue as follows.

Suppose that $(i^1, \ldots, i^{n-1})$ is a temporary list of size $n-1$ where $n \geq 3$. We construct a temporary list of size n. Let $s^k := \mu'(i^k)$ for each $k \in \{1, \ldots, n-1\}$. Since $(i^1, \ldots, i^{n-1})$ is a temporary list, $\nu(i^{n-1}) \ P_{i^{n-1}} \ \mu'(i^{n-1})$. Let $s^n := \nu(i^{n-1})$. Note that $s^{n-1} \neq s^n$. By *non-wastefulness* of μ', as $s^n \ P_{i^{n-1}} \ \mu'(i^{n-1})$, we have $|\mu'(s^n)| = q_{s^n}$. Then, there is $i^n \in \mu'(s^n)$ such that

$\nu(i^n) \neq \mu'(i^n)$ (Otherwise, as $|\mu'(s^n)| = q_{s^n}$, we have $\mu'(s^n) = \nu(s^n)$ and thus $i^{n-1} \in \nu(s^n) = \mu'(s^n)$. Thus, $\mu'(i^{n-1}) = s^n$ but $\mu'(i^{n-1}) \equiv s^{n-1}$, which contradicts $s^{n-1} \neq s^n$). Thus, as ν dominates μ' at P_I, $\nu(i^n)\ P_{i^n}\ \mu'(i^n)$. Also, as $i^n \in \mu'(s^n)$, $\nu(i^{n-1}) \equiv s^n = \mu'(i^n)$. Thus, $(i^1, \ldots, i^n)$ is a temporary list of size n. If $\nu(i^n) = \mu'(i^k)$ for some $k \in \{1, \ldots, n-1\}$, then the list $(i^k, \ldots, i^n)$ is an improvement cycle from μ' to ν. As the set of students is finite, we eventually obtain an improvement cycle from μ' to ν. □

Given an improvement cycle $(i^1, \ldots, i^n)$ from μ' to ν at P_I, matching μ'' is said to be **induced by an improvement cycle** if $\mu''(i) = \nu(i^k) \equiv \mu'(i^{k+1})$ when $i = i^k$ for some $k \in \{1, \ldots, n\}$, and $\mu''(i) = \mu'(i)$ otherwise.

Claim 2. Suppose that ν dominates a non-wasteful matching μ' at P_I and μ'' is a matching induced by an improvement cycle from μ' to ν. Then, (i) either $\nu = \mu''$ or ν dominates μ'' at P_I, and (ii) μ'' is non-wasteful at P_I.

Proof. The proof of Part (i) is straightforward and thus omitted. We prove Part (ii). Let the improvement cycle be $(i^1, \ldots, i^n)$. Let $i \in I$ and $s \in S$ be such that $s\ P_i\ \mu''(i)$. We need to show $|\mu''(s)| = q_s$. We consider two cases:

Case 1: for all $k \in \{1, \ldots, n\}$, $i \neq i^k$. Then $\mu''(i) = \mu'(i)$. Thus, as $s\ P_i\ \mu''(i)$, we have $s\ P_i\ \mu'(i)$. By *non-wastefulness* of μ', $|\mu'(s)| = q_s$. Since by construction $|\mu'(s)| = |\mu''(s)|$, we have$|\mu''(s)| = q_s$.

Case 2: for some $k \in \{1, \ldots, n\}$, $i = i^k$. Then $\nu(i^k)\ P_{i^k}\ \mu'(i^k)$ and $\nu(i^k) = \mu''(i^k)$. Thus, as $s\ P_i\ \mu''(i)$, $s\ P_i\ \nu(i)\ P_i\ \mu'(i)$. Then, by *non-wastefulness* of μ', $|\mu'(s)| = q_s$. Since by construction $|\mu'(s)| = |\mu''(s)|$, we have$|\mu''(s)| = q_s$. □

We say that **matching ν can be achieved from matching μ by improvement cycles at** P_I if there is a partition $\{J^r\}_{r=1}^R$ of J such that each subset in the partition forms an improvement cycle from μ to ν at P_I.

Claim 3. Given that matching ν dominates the non-wasteful matching μ at P_I, ν can be achieved from μ by improvement cycles at P_I.

Proof. We shall reach ν from μ iteratively via the following **improvement cycles algorithm**. Let $\mu^0 = \mu$. Let $r \geq 1$.

Step r: From the previous step, $\mu^{r-1} \neq \nu$ is non-wasteful and ν dominates μ^{r-1} at P_I. Thus, by Claim 1, there is an improvement cycle from μ^{r-1} to ν. Choose one such cycle. Let μ^r be the matching induced by the improvement cycle. By Claim 2-Part (i), either $\mu^r = \nu$ or ν dominates μ^r at P. If the former happens, the algorithm stops. Otherwise, go to the next step. Note that by Claim 2-Part (ii), μ^r is non-wasteful at P_I and ν dominates μ^r at P_I.

Let $M^r := \{i \in I \mid \nu(i) \neq \mu^r(i)\}$ and J^r be the set of students involved in the improvement cycle in step r. Note that $J^r \neq \emptyset$. Then, $M^r = M^{r-1} \setminus J^r = M^0 \setminus (\cup_{t=1}^r J^r)$. Thus, as the set of students is finite, the algorithm stops in a finite step.

Now we are ready to prove Claim 3. Note that in each step of the algorithm, an improvement cycle from μ^{r-1} to ν is also an improvement cycle from μ to ν. Clearly, $\{J^r\}_{r=1}^R$ is a partition of J, where R is the last step of the algorithm. □

Proof of Lemma 2

Suppose that $\nu(i) \neq DA_i^\tau(P_I, R_S)$. Since DA^τ is non-wasteful, it follows from Claim 3 that student i is in some subset J' in the partition of J and we can order elements of J' to form an improvement cycle from $DA^\tau(P_I, R_S)$ to ν. Let $(i^1, \dots, i^n)$ be that cycle. Without loss of generality, let $i^1 = i$. Then, by the definition of an improvement cycle, $\nu(i^n) = DA^\tau_{i^{n+1}}(P_I, R_S) \equiv DA^\tau_{i^1}(P_I, R_S)$ and $\nu(i^n)\ P_{i^n}\ DA^\tau_{i^n}(P_I, R_S)$. Thus, $DA^\tau_{i^1}(P_I, R_S)\ P_{i^n}\ DA^\tau_{i^n}(P_I, R_S)$. Hence, $DA^\tau_{i^1}(P_I, R_S) \equiv DA^\tau_i(P_I, R_S)$, which is overdemanded. ■

A.3 Proof of Lemma 3

We start with the following useful claim whose straightforward proof is omitted.

Claim 4. Let T be the set of all schools that are underdemanded at $(P_I, R_S; \tau)$. For all R'_T, (i) $DA^\tau(P_I, R_S) = DA^\tau(P_I, R'_T, R_{-T})$, and (ii) if s is underdemanded at $(P_I, R_S; \tau)$, then s is also underdemanded at $(P_I, R'_T, R_{-T}; \tau)$.

Suppose that under DA student i is assigned school s^* that is overdemanded at $(P_I, R_S; \tau)$. Let S^u be the set of all schools that are underdemanded at $(P_I, R_S; \tau)$. Note that $S^u \neq \emptyset$ by Lemma 1, and any underdemanded school $s \in S^u$ is strictly worse than s^* in P_i as s^* is overdemanded. We consider two cases:

P_i	P'_i	P''_i
$P_i\|_{U(P_i,s^*)\setminus\{s^*\}}$	$P_i\|_{U(P_i,s^*)\setminus\{s^*\}}$	$P_i\|_{U(P_i,s^*)\setminus\{s^*\}}$
s^*	all underdemanded schools at $(P_I, R_S; \tau)$	s^u
$\vdots$	s^*	s^*
	$\vdots$	$\vdots$

Case 1: for some $s \in S^u$, $|DA^\tau_s(P_I, R_S)| < q_s$. Let $s^u \in S^u$ be one such school, i.e., $|DA^\tau_{s^u}(P_I, R_S; \tau)| < q_{s^u}$. Consider the preference P''_i that upgrades s^u above s^* as described in the above table. Let $P''_I := (P''_i, P_{-i})$. Let μ'' be a matching such that $\mu''(i) = s^u$ and for all $j \neq i$, $\mu''(j) = DA^\tau_j(P_I, R_S)$. Then, μ'' is stable at (P''_I, R^τ_S). For problem (P''_I, R_S, τ) DA works in exactly the same way as it does for (P_I, R_S, τ) until right before the step i applies to s^u. Hence, $DA^\tau_i(P''_I, R_S) \notin U(P_i, s^*) \setminus \{s^*\}$. Note that matching $DA^\tau(P''_I, R_S)$ dominates at P''_I any matching that is stable at (P''_I, R^τ_S); μ'' is stable at (P''_I, R^τ_S); and $\mu''(i) = s^u$. Thus, $DA^\tau_i(P''_I, R_S) = s^u$. Also, for each student $j \neq i$, $DA^\tau_j(P''_I, R_S)\ R_j\ \mu''(j)$. Thus, for all $j \neq i$, as $s^u \in S^u$ and $\mu''(j) = DA^\tau_j(P_I, R_S)$, we have $DA^\tau_j(P''_I, R_S)\ R_j\ DA^\tau_j(P_I, R_S)\ R_j\ s^u$. This implies that s^u is underdemanded at $(P''_I, R_S; \tau)$.

Case 2: for all $s \in S^u$, $|DA^\tau_s(P_I, R_S)| = q_s$. As s^* is overdemanded at $(P_I, R_S; \tau)$, all schools in $U(P_i, s^*)$ are overdemanded at $(P_I, R_S; \tau)$. Thus, since $S^u \neq \emptyset$ by Lemma 1, all schools in S^u are less desirable than s^* to student i in P_i. We consider the preference P'_i that upgrades all underdemanded schools in S^u above s^* as described in the above table. Moreover, for all $s \in S^u$, let R'_s be the priority such that i has the lowest priority for s, and the relative rankings of all other students is the same as in R_s. Let $P'_I := (P'_i, P_{-i})$ and $R'_S := (R'_{S^u}, R_{-S^u})$.

We first show that $DA^\tau_i(P'_I, R'_S) \in S^u$. Suppose not. Then, under DA all schools in S^u reject i at $(P'_I, R'_S; \tau)$. Since i has the lowest priority at all schools in S^u and $DA^\tau(P_I, R_S)$ is stable at (P_I, R^τ_S), $DA^\tau(P_I, R_S)$ is also stable at (P'_I, R'_S). Then, as the DA matching is student-optimal stable, $DA^\tau(P'_I, R'_S) = DA^\tau(P_I, R_S)$. Thus, all overdemanded schools at $(P_I, R_S; \tau)$ are still overdemanded at $(P'_I, R'_S; \tau)$, and all underdemanded schools at $(P_I, R_S; \tau)$ become overdemanded at $(P'_I, R'_S; \tau)$. Then, all schools are overdemanded at $(P'_I, R'_S; \tau)$, which contradicts Lemma 1.

Now, letting $s^u := DA^\tau_i(P'_I, R'_S) \in S^u$, consider the preference P''_i that upgrades s^u above s^* in P_i as described in the above table. Let $P''_I := (P''_i, P_{-i})$. We will show that $DA^\tau_i(P''_I, R'_S) = s^u$. For $(P''_I, R'_S; \tau)$ DA works in the same way as for $(P_I, R_S; \tau)$ until student i applies to s^u. Thus, $DA^\tau_i(P''_I, R'_S) \notin U(P''_i, s^u) \setminus \{s^u\}$. Then, by *strategy-proofness* of DA^τ, $DA^\tau_i(P''_I, R'_S)$ R''_i $DA^\tau_i(P'_I, R'_S)$, i.e., $DA^\tau_i(P''_I, R'_S)$ R''_i s^u. Thus, $DA^\tau_i(P''_I, R'_S) = s^u$.

It remains to show that $DA^\tau_i(P''_I, R_S) = s^u$ and s^u is underdemanded at $(P''_I, R_S; \tau)$. Note that $DA^\tau(P_I, R_S)$ is stable at (P''_I, R'_S), since $DA^\tau(P_I, R_S)$ is stable at (P_I, R^τ_S) and i has the lowest priority for s^u with $|DA^\tau_{s^u}(P_I, R_S)| = q_{s^u}$. Since matching $DA^\tau(P''_I, R'_S)$ is student-optimal stable, we have for all $j \neq i$ and all $s \in S^u$, $DA^\tau_j(P''_I, R'_S)$ R''_j $DA^\tau_j(P_I, R_S)$ R_j s and $DA^\tau_i(P''_I, R'_S)$ R''_i s^u R''_i s. Since $P''_{-i} = P_{-i}$, for all $j \in I$ and all $s \in S^u$, $DA^\tau_j(P''_I, R'_S)$ R''_j s. That is, all schools in S^u, including s^u, are underdemanded at $(P''_I, R'_S; \tau)$. Hence, by Claim 4, $DA^\tau_i(P''_I, R_S) = DA^\tau_i(P''_I, R'_S) = s^u$, which is underdemanded at $(P''_I, R_S; \tau)$. ■

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Proof of strategy-proofness of the mechanism φ in Theorem 1

We show that for all P_I, R_S $i \in I$, and P'_i, student i cannot manipulate φ via P'_i at P_I, i.e., $\varphi_i(P_I, R_S)\ R_i\ \varphi_i(P'_i, P_{-i}, R_S)$. Fix P_I, R_S, $i \in I$, and P'_i. If $R_S \neq \bar{R}_S$ or $[R_S = \bar{R}_S$ and $i \in I \setminus \{i_1, i_2\}]$, then for all P''_I, $\varphi_i(P''_I, R_S) = DA^\tau_i(P''_I, R_S)$ and DA^τ is strategy-proof. Thus, student i cannot manipulate φ via any P'_i at (P_I, R_S).

Suppose that $R_S = \bar{R}_S$ and $i = i_1$.

<u>Case 1-i:</u> $P_{I\setminus\{i_1\}} \notin (\mathcal{P}^{**}_{I_1\setminus\{i_1\}}, \mathcal{P}^*_{I_2}, \mathcal{P}^3_{I_3}, \ldots, \mathcal{P}^m_{I_m}, \mathcal{P}^*_{I_{m+1}})$. Then, for all P''_{i_1}, $\varphi(P''_{i_1}, P_{-i_1}, R_S) = DA^\tau(P''_{i_1}, P_{-i_1}, R_S)$ and DA^τ is strategy-proof. Thus, student i_1 cannot manipuate φ via P'_{i_1} at (P_I, R_S).

<u>Case 1-ii:</u> $P_{I\setminus\{i_1\}} \in (\mathcal{P}^{**}_{I_1\setminus\{i_1\}}, \mathcal{P}^*_{I_2}, \mathcal{P}^3_{I_3}, \ldots, \mathcal{P}^m_{I_m}, \mathcal{P}^*_{I_{m+1}})$. We first show that

$$\text{for all } P''_{i_1}, DA^\tau_{i_1}(P''_{i_1}, P_{-i_1}, R_S) = t_1. \tag{2}$$

If $P''_{i_1} \in \mathcal{P}^{**}$, then by (1) $DA^\tau_{i_1}(P''_{i_1}, P_{-i_1}, R_S) = t_1$. Suppose $P''_{i_1} \in \mathcal{P}^*$. Then, since $|\sum_{t\in T} q_t| < |I|$, by the prefence profile and non-wastefulness of DA^τ, each school $t \in T$ is fully assigned $|q_t|$

students under DA. Moreover, schools $t_3, \dots, t_m$ are filled by students in $I_3, \dots, I_m$ under DA at $(P''_{i_1}, P_{-i_1}, R_S)$. Thus, i_1 is assigned t_1, t_2, or some $s \in S \setminus T$. Suppose on the contrary that i_1 is assigned t_2 or $s \in S \setminus T$. Then, s_1 is assigned to some student in $I_2 \cup I_{m+1}$. However, (i_1, t_1) blocks the DA matching, which contradicts the stability of DA^τ. Thus, $DA^\tau_{i_1}(P''_{i_1}, P_{-i_1}, R_S) = t_1$. This completes the proof of (2).

Suppose $P_{i_1} \in \mathcal{P}^*$. Then it follows from (2) and the construction of φ that $\varphi_{i_1}(P_I, R_S) = DA^\tau_{i_1}(P_I, R_S) = t_1$, $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, R_S) = t_1$ if $P'_{i_1} \in \mathcal{P}^*$, and $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, R_S) = t_2$ if $P'_{i_1} \in \mathcal{P}^{**}$. In any case, as $P_{i_1} \in \mathcal{P}^*$, $\varphi_{i_1}(P_I, R_S)\ R_{i_1}\ \varphi_{i_1}(P'_{i_1}, P_{-i_1}, R_S)$. On the other hand, suppose $P_{i_1} \in \mathcal{P}^{**}$. Then it follows from (2) and the construction of φ that $\varphi_{i_1}(P_I, R_S) = t_2$, $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, R_S) = DA^\tau_{i_1}(P'_{i_1}, P_{-i_1}, R_S) = t_1$ if $P'_{i_1} \in \mathcal{P}^*$, and $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, R_S) = t_2$ if $P'_{i_1} \in \mathcal{P}^{**}$. Thus, in any case, as $P_{i_1} \in \mathcal{P}^{**}$, $\varphi_{i_1}(P_I, R_S)\ R_{i_1}\ \varphi_{i_1}(P'_{i_1}, P_{-i_1}, R_S)$. Therefore, i_1 cannot manipuate φ via P'_{i_1} at (P_I, R_S).

We finally consider the case where $R_S = \bar{R}_S$ and $i = i_2$.

Case 2-i: $P_{I\setminus\{i_2\}} \notin (\mathcal{P}^{**}_{I_1}, \mathcal{P}^*_{I_2\setminus\{i_2\}}, \mathcal{P}^3_{I_3}, \dots, \mathcal{P}^m_{I_m}, \mathcal{P}^*_{I_{m+1}})$. Then, for all P''_{i_2}, $\varphi(P''_{i_2}, P_{-i_2}, R_S) = DA^\tau(P''_{i_2}, P_{-i_2}, R_S)$ and DA^τ is strategy-proof. Thus, student i_2 cannot manipuate φ via P'_{i_2} at (P_I, R_S).

Case 2-ii: $P_{I\setminus\{i_2\}} \in (\mathcal{P}^{**}_{I_1}, \mathcal{P}^*_{I_2\setminus\{i_2\}}, \mathcal{P}^3_{I_3}, \dots, \mathcal{P}^m_{I_m}, \mathcal{P}^*_{I_{m+1}})$. We show that

$$\text{for all } P''_{i_2}, DA^\tau_{i_2}(P''_{i_2}, P_{-i_2}, R_S) = t_2. \tag{3}$$

If $P''_{i_2} \in \mathcal{P}^*$, then by (1) $DA^\tau_{i_2}(P''_{i_2}, P_{-i_2}, R_S) = t_2$. Suppose $P''_{i_2} \in \mathcal{P}^{**}$. Then, since $|\sum_{t\in T} q_t| < |I|$, by the prefernce profile and non-wastefulness of DA^τ, each school $t \in T$ are fully assigned $|q_t|$ students under DA. Moreover, schools $t_3, \dots, t_m$ are filled by students in $I_3, \dots, I_m$ under DA at $(P''_{i_2}, P_{-i_2}, R_S)$. Thus, i_1 is assigned t_1, t_2, or some $s \in S \setminus T$ under DA. Suppose on the contrary that i_2 is assigned t_1 or $s \in S \setminus T$. Then, t_2 is assigned to some student in $I_1 \cup I_{m+1}$. However, (i_2, t_2) blocks the DA matching, which contradicts the stability of DA^τ. Thus, $DA^\tau_{i_2}(P''_{i_2}, P_{-i_2}, R_S) = t_2$. This completes the proof of (3).

Suppose $P_{i_2} \in \mathcal{P}^*$. Then it follows from (3) and the construction of φ that $\varphi_{i_2}(P_I, R_S) = t_1$, $\varphi_{i_2}(P'_{i_2}, P_{-i_2}, R_S) = t_1$ if $P'_{i_2} \in \mathcal{P}^*$, and $\varphi_{i_2}(P'_{i_2}, P_{-i_2}, R_S) = DA^\tau_{i_2}(P'_{i_2}, P_{-i_2}, R_S) = t_2$ if $P'_{i_2} \in \mathcal{P}^{**}$. In any case, since $P_{i_2} \in \mathcal{P}^*$, $\varphi_{i_2}(P_I, R_S)\ R_{i_2}\ \varphi_{i_2}(P'_{i_2}, P_{-i_2}, R_S)$. On the other hand, suppose $P_{i_2} \in \mathcal{P}^{**}$. Then it follows from (3) and the construction of φ that $\varphi_{i_2}(P_I, R_S) = DA^\tau_{i_2}(P_I, R_S) = t_2$, $\varphi_{i_2}(P'_{i_2}, P_{-i_2}, R_S) = t_1$ if $P'_{i_2} \in \mathcal{P}^*$, and $\varphi_{i_2}(P'_{i_2}, P_{-i_2}, R_S) = DA^\tau_{i_2}(P'_{i_2}, P_{-i_2}, R_S) = t_2$ if $P'_{i_2} \in \mathcal{P}^{**}$. Thus, in any case, since $P_{i_2} \in \mathcal{P}^{**}$, $\varphi_{i_2}(P_I, R_S)\ R_{i_2}\ \varphi_{i_2}(P'_{i_2}, P_{-i_2}, R_S)$. Therefore, i_2 cannot manipulate φ at (P_I, R_S). ■

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