WZB
WISSENSCHAFTSZENTRUM BERLIN
FÜR SOZIALFORSCHUNG
SOCIAL SCIENCE RESEARCH
CENTER BERLIN

discussion papers

FS IV 02 – 22

On the Profitability of Collusion in Location Games

Steffen Huck *
Vicki Knoblauch **
Wieland Müller ***

*   University College London and ELSE
**   University of Connecticut
*** New York University and Humboldt University

December 2002

ISSN Nr. 0722 - 6748

Forschungsschwerpunkt
Markt und politische Ökonomie

Research Area
Markets and Political Economy

Zitierweise/Citation:

Steffen Huck, Vicki Knoblauch, Wieland Müller, **On the Profitability of Collusion in Location Games**, Discussion Paper FS IV 02-22, Wissenschaftszentrum Berlin, 2002.

Wissenschaftszentrum Berlin für Sozialforschung gGmbH,
Reichpietschufer 50, 10785 Berlin, Tel. (030) 2 54 91 – 0
Internet: www.wz-berlin.de

ABSTRACT

On the Profitability of Collusion in Location Games

by Steffen Huck, Vicki Knoblauch and Wieland Müller*

In this note we take a first step towards the analysis of collusion in markets with spatial competition, focusing on the case of pure location choices. We find that collusion can only be profitable if a coalition contains more than half of all players. This result holds for location games played in *k*-dimensional Euclidean space as long as consumers are distributed via atomless density functions. For competition on the unit interval, unit circle, and unit square we also derive sufficient conditions for collusion to be profitable.

ZUSAMMENFASSUNG

Zur Profitabilität von Kollusion in Standortspielen

Wir untersuchen Kollusion in Märkten, in denen die einzige strategische Variable der Akteure ihre Ortswahl ist. Für Spiele in *k*-dimensionalen Euklidischen Räumen mit massepunktfreien Verteilungen zeigen wir, dass Kollusion nur profitabel sein kann, wenn wenigstens die Hälfte aller Akteure kolludieren. Für Wettbewerb auf dem Einheitsintervall, dem Einheitskreis und dem Einheitsquadrat etablieren wir hinreichende Bedingungen für die Profitabilität von Kollusion.

* We are indebted to Kai Konrad and an anonymous referee for many helpful comments.

1 Introduction

While the economics literature has paid considerable attention to collusion in Bertrand and Cournot markets, collusion with different sorts of competition has been largely neglected. In this note we take a first step towards the analysis of collusion in pure location games as introduced by Hotelling (1929). Such models capture competition in many important industries where price competition is not feasible, for example, because of regulation (as in the case of pharmacies) or vertical restraints (as in the case of book sellers).[1]

Our results are based on an approach that relies on rather weak rationality requirements. In particular, we do not solve the non-cooperative game in which some of the players can reach binding agreements. Instead, we simply require that players will only decide to collude if they can guarantee themselves a payoff better than the payoff expected "behind the veil of ignorance". The reason for this approach is simple: it is as we will see extremely difficult to find Nash equilibria for location games with collusion. We argue that in the absence of reliable non-cooperative solutions players should be conservative and only collude if they know for sure that this will be profitable. Accordingly, our definition of profitability relies on a maxmin approach. Nevertheless, we include one section on Nash equilibrium where we show that in some cases the non-cooperative solution coincides with ours.

For linear and circular cities with a uniform distribution of consumers we find that collusion is profitable if and only if more than half of the players collude. Part of this result can be generalized to location games in multi-dimensional spaces with arbitrary density functions: As long as the distribution of consumers is atomless, collusion can only be profitable if more than half of all firms cooperate. For competition on the unit interval, unit circle, and unit square we are also able to derive sufficient conditions for collusion to be profitable. These results are of considerable relevance for the topic of merger in markets with limited price competition.

The remainder of the paper is organized as follows. Section 2 introduces the general setup and notation. Section 3 deals with the simplest one-dimensional cases, i.e., with linear and circular cities with uniform consumer densities. Section 4 deals with the general multi–dimensional case and establishes the main theorem of the paper. Section 5 adds sufficient conditions for collusion to be profitable in games on the unit line, unit circle and unit square. Section 6 discusses Nash equilibria for location games with collusion and Section 7 concludes.

[1] They can also be applied to parliamentary elections that are not winner-take-all contests.

2 Setup and definitions

Let $\Gamma(O;P)$ be a location game on $O \subseteq \mathbb{R}^k$ with set of players P. Let $p^i \in P$ be player i with $i = 1,2,\ldots,n$. Each player p^i chooses a location $x^i \in O$. Consumers are distributed over O via a Lebesgue measurable density function f with total mass 1. Let $d(o,o')$ be the distance between two points $o,o' \in O$. Each consumer is assumed to buy one unit of an unspecified good from the player closest to her. That is, a consumer at $o \in O$ buys from player p^i only if $d(o,x^i) = \min_j d(o,x^j)$. If there are more than one closest player then the consumer is assumed to buy from each closest player with the same probability. The price of the good is fixed at 1 and production costs are normalized to zero.

Let $O^i(\Gamma) = \left\{ o \mid d(o,x^i) = \min_j d(o,x^j) \right\}$. Player p^i's market share and profit is then given by $\pi^i(\Gamma) = \frac{1}{r^i}\int_{O^i} f(o)do$ where r^i denotes the number of players located at x^i. By assumption, $\sum_i \pi^i = 1$. By virtue of this fact, we say that a player's expected payoff before the game is actually played ("behind the veil of ignorance") is $\frac{1}{n}$.[2]

Next we define for integer m with $1 \leq m < n$ a set $V(m)$ of reals with $v \in V(m)$ if there is a collusion strategy for a set $\mathcal{M} \subseteq P$ of m players that guarantees them a total payoff of at least v. Let $v(m) = \sup V(m)$.[3]

Definition 1 Collusion of a set of m players is profitable if $v(m) > \frac{m}{n}$.

3 The one–dimensional case with uniform distributions

3.1 Linear cities

Let us first consider the standard textbook case of a "linear city" in which $O = [0,1]$ and in which consumers are uniformly distributed. How can a set of m players guarantee itself a "high" payoff? Suppose $m > n - m$, i.e., suppose that more than half of all firms are in the set of colluding players. In that case the colluding players can minimize the payoff obtainable to a firm outside the coalition by "evenly spreading out." If f is uniform, the firms in the set can guarantee themselves a payoff of $\frac{3m-n}{2m}$ by locating themselves at $(k, 3k, 5k, \ldots, 1-k)$ with $k = \frac{1}{2m}$. To see this, note that in this case a firm outside the set $\mathcal{M}$ is indifferent between all possible locations as each location yields a payoff of $\frac{1}{2m}$. Furthermore, the worst thing that can happen to the players in $\mathcal{M}$ is that the firms

[2] For example, a player could expect that every assignment of players to equilibrium locations is equally likely.

[3] In other words, $v(m)$ is the maxmin payoff of the coalition.

outside locate in different intervals, say, one between k and 3k, one between 3k and 5k and so on. If they do, the players in $\mathcal{M}$ earn $1 - \frac{n-m}{2m} = \frac{3m-n}{2m}$. And as this is larger than $\frac{m}{n}$ for $m > \frac{n}{2}$ collusion turns out to be profitable. Thus $m > \frac{n}{2}$ is sufficient for collusion to be profitable in linear cities with a uniform distribution of consumers. That it is also necessary in this case is stated in

Proposition 1 In linear cities with a uniform distribution of consumers collusion pays if and only if $m > \frac{n}{2}$.

Proof The argument above shows that $m > \frac{n}{2} \Rightarrow v(m) > \frac{m}{n}$. Next observe that, by definition,

$$v(m) + v(n-m) \leq 1. \tag{1}$$

Hence, $m = \frac{n}{2} \Rightarrow m = n - m \Rightarrow v(m) \leq \frac{1}{2} = \frac{m}{n}$, i.e., collusion is not profitable if exactly half of all firms cooperate. The proof is completed by showing that collusion is also not profitable if $m < \frac{n}{2}$: If $1 \leq m < \frac{n}{2}$, then $\frac{n}{2} < n - m \leq n - 1$ so that by the first part of the proof $v(n-m) > \frac{n-m}{n}$. Therefore, by (1) $v(m) < 1 - \frac{n-m}{n} = \frac{m}{n}$. ¤

3.2 Circular cities

A further popular space to study location games on is a circle. In contrast to the line a set of m colluding firms can divide a circle into at most m arcs as opposed to m + 1 segments on the line. Nevertheless, one obtains the identical condition for collusion to be profitable.

Proposition 2 In circular cities with a uniform distribution of consumers collusion pays if and only if $m > \frac{n}{2}$.

Proof Position the colluding firms such that there are m arcs with mass $\frac{1}{m}$ each. If $m \geq \frac{n}{2}$ the maximum total payoff the non–colluding firms can obtain is $\frac{n-m}{2m}$, i.e., by using this strategy the colluding firms can ensure a payoff of $\frac{3m-n}{2m}$ which is greater than $\frac{m}{n}$ if $m > \frac{n}{2}$. Using (1) again completes the proof. ¤

4 The multi–dimensional case

The following result is the main result of the paper. It generalizes one of the two insights gained above, namely that collusion in location games can only be profitable if more than half of all firms cooperate. This result holds for arbitrary bounded open subsets of R^k and for arbitrary bounded atomless density functions.

Theorem 1 Suppose consumers are distributed over a bounded open subset $O \subseteq R^k$ via a bounded Lebesgue measurable density function f of total mass 1. For the n–player location game $\Gamma(O; P)$ it is not profitable for a set of m players to collude if $m \leq \frac{n}{2}$.

Proof Suppose the m colluding players $p^1; p^2; \ldots; p^m$ locate at $x^1; x^2; \ldots; x^m \in O$, not necessarily distinct.

Case 1. $n - m \geq 2m$. Then for each i, $1 \leq i \leq m$, let p^{m+2i-1} and p^{m+2i} locate at x^{m+2i-1} and x^{m+2i}, two points ε units apart on a line through x^i, with x^i between x^{m+2i-1} and x^{m+2i} and ε chosen as follows: Let B be a k–dimensional ball containing O and let A be the $k-1$–dimensional volume of the $k-1$–dimensional disk formed by intersecting B with a hyperplane through its center. Choose ε such that $\varepsilon < \frac{1}{nA \sup f}$ and such that ε is small enough to guarantee $x^{2m+i-1}; x^{2m+i} \in O$ for $1 \leq i \leq m$. Let the rest of the non–colluding players, $p^{3m+1}; p^{3m+2}; \ldots; p^n$ locate anywhere in O. Since the consumers won by p^i, $1 \leq i \leq m$, lie between two hyperplanes ε units apart, π^i is at most $\varepsilon A \sup f < \frac{1}{n}$. Hence, $v(m) < \frac{m}{n}$.

Case 2. $m < n - m < 2m$. For $1 \leq i \leq m$, define the provisional market set $O^i_{prov} = O^i(\Gamma')$ with $\Gamma' = \Gamma(O; \mathcal{M})$, i.e., O^i_{prov} contains the points in O that are nearer to x^i than to any other $x^j \neq x^i$ with both $i; j \leq m$. Accordingly, define the provisional payoff $\pi^i_{prov} = \pi^i(\Gamma')$. W.l.o.g. assume that the sequence $\pi^1_{prov}; \pi^2_{prov}; \ldots; \pi^m_{prov}$ is non–decreasing. Now locate $3m - n$ of the non–colluding players at $x^1; x^2; \ldots; x^{3m-n}$ and use the remaining $2n - 4m$ players to bracket $x^{3m-n+1}; x^{3m-n+2}; \ldots; x^m$ as in case 1, but do net yet choose ε. Notice that (i) $3m - n > 0$; (ii) $2n - 4m > 0$; (iii) $(3m - n) + (2n - 4m) = n - m$; and (iv) $(3m - n) + (2n - 4m)/2 = m$. Since the sequence $\pi^1_{prov}; \pi^2_{prov}; \ldots; \pi^m_{prov}$ is non–decreasing, the sum of the provisional payoffs $\pi^1_{prov} + \pi^2_{prov} + \ldots + \pi^{3m-n}_{prov}$ is at most $\frac{3m-n}{m}$. Therefore, the final total payoffs to the colluding players $\sum_{i=1}^{m} \pi^i$ is at most $\frac{3m-n}{2m} + \varepsilon(n - 2m)A \sup f$. Now notice that $\frac{3m-n}{2m} < \frac{m}{n}$. Hence, it is possible to choose ε such that $\frac{m}{n} - \sum_{i=1}^{m} \pi^i > 0$. Collusion is not profitable.

Case 3. $m = n - m$. Nonprofitability follows from (1) as in the proof of Proposition 1. □

Thus, we know that collusion in location games (on bounded open subsets of R^k in which consumers are distributed via atomless density functions) can only be profitable if more than half of all firms join the set $\mathcal{M}$.

Remark 1 Note that neither the closed interval $[0; 1]$ nor a circle is an open subset of an Euclidean space. However, the conclusion of the theorem holds for location games on these sets, since the techniques of the proof apply. More particularly, it is possible to bracket colluding players as in the proofs. In fact, a colluding player at 0 or 1 in $[0;1]$ can be bracketed by a single non–colluding player.

Remark 2 The theorem concerns location games de…ned using Euclidean distances, i.e., straight line distances. Implicitly, this means that consumers may travel along routes that do not belong to O. However, the theorem applies, for example, to a circle (or rather the conclusion of the theorem holds—see Remark 1) even when the distance between two points is the length of the arc joining them, since for a circle in R^2 a consumer's nearest player is the same whether distance is de…ned as Euclidean distance or as arc length.

The theorem disallows atoms of consumers. The following example demonstrates the necessity of this assumption.

Example Consider the 5–player location game on $[0;1]$ with two consumers, one at $\frac{1}{4}$ and one at $\frac{2}{3}$. Suppose p^1 and p^2 collude by locating at $\frac{1}{4}$ and $\frac{2}{3}$ respectively. Their worst total payo¤ occurs when p^3 and p^4 locate at $\frac{1}{4}$ and p^5 locates at $\frac{2}{3}$. The total payo¤ of p^1 and p^2 is then $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ which is greater than the veil of ignorance expected payo¤ of $2(\frac{2}{5}) = \frac{4}{5}$. Collusion is pro…table with $m = 2$ even though $m < \frac{n}{2}$. As in the proof of Proposition 1, where it is shown that the complement of a pro…table set of colluding players is unpro…table, collusion is unpro…table for $m = 3$, even though in that case $m > \frac{n}{2}$. ¤

5 Su¢cient conditions for unit interval, unit circle, and unit square

The main theorem above showed that $m > \frac{n}{2}$ is necessary for collusion to be successful. In the following we will establish su¢cient conditions for collusion to be pro…table in a location game played on the unit interval, the unit circle, and the unit square. Notice that in each case the solution prescribes that the colluding players behave according to the above identi…ed strategies, i.e., they will evenly spread out making other players indi¤erent between locations.

Proposition 3 In linear cities, collusion pays if $\frac{\sup f}{\inf f} < \frac{2m}{n}$.

Remark 3 Note that $\sup f = \inf f \geq 1$. Thus, the condition in Proposition 3 ensures that $m > n=2$.

Proof of Proposition 3 W.l.o.g. let $x^1 \leq x^2 \leq \cdots \leq x^m$ be the set of locations occupied by the colluding players chosen so that

$$\int_0^{x^1} f(o)do = \frac{1}{2}\int_{x^1}^{x^2} f(o)do = \frac{1}{2}\int_{x^2}^{x^3} f(o)do = \cdots = \int_{x^m}^{1} f(o)do = \frac{1}{2m}.$$

If a non–colluding player locates to the left of x^1 or to the right of x^m, his payoff is at most $\frac{1}{2m} < \frac{1}{n}$. If a non–colluding player locates between x^i and x^{i+1}, his payoff is $\int_c^d f(o)do$ where $x^i < c < d < x^{i+1}$ and $d - c = \frac{1}{2}(x^{i+1} - x^i)$. Then

$$\begin{aligned}
\int_c^d f(o)do &\leq (d-c)\sup f \\
&= \frac{x^{i+1}-x^i}{2}\inf f\left(\frac{\sup f}{\inf f}\right) \\
&\leq \frac{1}{2}\int_{x^i}^{x^{i+1}} f(o)do\left(\frac{\sup f}{\inf f}\right) \\
&< \frac{1}{2m}\left(\frac{2m}{n}\right) \\
&= \frac{1}{n}.
\end{aligned}$$

If a non–colluding player locates at x^i, $1 \leq i \leq m$, then he shares the market set O^i with p^i. By the argument above, the portion of O^i to the left of x^i has consumer mass less than $\frac{1}{n}$, as does the portion of O^i to the right of x^i. Therefore, the payoff to each non–colluding player is less then $(\frac{1}{n} + \frac{1}{n})/2 = \frac{1}{n}$. Since in all these cases the payoff to a non–colluding player is less than $\frac{1}{n}$; the total payoff to the colluding players is more than $1 - \frac{n-m}{n} = \frac{m}{n}$. Collusion is profitable. ¤

The sufficient condition in Proposition 3 is stronger than necessary. For instance, we used as an assumption on f only that $\frac{\sup\{f(x):x^i<x<x^{i+1}\}}{\inf\{f(x):x^i<x<x^{i+1}\}} < \frac{2m}{n}$. This allows any amount of variation to the left of x^1 and to the right of x^m and, if m is large, between x^1 and x^m.[4]

Proposition 4 In circular cities, collusion pays if $m > \frac{n}{2}$ and $\frac{\sup f}{\inf f} < \frac{2m}{n}$.

Proof Analogous to the proofs of Propositions 2 and 3. ¤

[4] Moreover, the firms located at x^1 and x^m could move further into the interior as the mass on the fringes has only to be smaller than $\frac{1}{n}$. Using this, one can increase the allowed variation between x^1 and x^m from $\frac{2m}{n}$ to $\left[\frac{2(m-1)}{n-2}\right]^{m-1} > \frac{2m}{n}$. To see this, simply observe that the colluding players can position themselves so that the remaining mass between x^1 and x^m, $1 - \frac{2}{n}$, is equally distributed over $m - 1$ intervals. The proof then goes through with $\frac{\sup\{f(x):x^i<x<x^{i+1}\}}{\inf\{f(x):x^i<x<x^{i+1}\}} < \frac{2(m-1)}{n-2}$. Therefore, $\frac{\sup\{f(x):x^1<x<x^m\}}{\inf\{f(x):x^1<x<x^m\}}$ can be as large as $\left[\frac{2(m-1)}{n-2}\right]^{m-1}$.

Finally, we look at location games played on the unit square with uniform consumer density.

Proposition 5 For the n–player location game on the square $[0;1] \times [0;1]$ with consumers distributed uniformly, collusion is pro…table if there is a positive integer h with $(2h+1)^2 - h^2 \leq m < n < (2h+1)^2$.

Proof Suppose $m; n$ and h satisfy the hypotheses of the theorem. Consider the set C of points in $[0;1] \times [0;1]$ of the form $\left(\frac{i-1/2}{2h+1}; \frac{j-1/2}{2h+1}\right)$ where i and j are integers, $1 \leq i; j \leq 2h+1$, and i and j are not both even. There are exactly $(2h+1)^2 - h^2$ points in C. Locate the m colluding players so that there is at least one of them at each point of C (recall that $m \geq (2h+1)^2 - h^2$). In the course of proving that an in…nite square lattice is a Nash equilibrium for the location game in the plane with consumers distributed uniformly, Knoblauch (2002) proved that in the location game on $[0;1] \times [0;1]$, any player with at least one opponent at each point of C earns a payo¤ of at most $\frac{1}{(2h+1)^2}$ so that the non–colluding players' total payo¤ is at most $\frac{n-m}{(2h+1)^2} < \frac{n-m}{n}$. Hence, $v(m) > \frac{m}{n}$. ¤

For large n, the proposition says, roughly, that collusion is pro…table if $m > \frac{3n}{4}$. This interpretation follows from the fact that for large n there is an integer h such that $n < (2h+1)^2$, $\frac{(2h+1)^2}{n} \approx 1$, and $\frac{(2h+1)^2 - h^2}{n} \approx \frac{3}{4}$. For example, if $n = 1;000;000$ choose $h = 500$. Then $(2h+1)^2 = 1;002;001$ and $(2h+1)^2 - h^2 = 752;001$. The proposition says collusion is pro…table if $\frac{m}{1;000;000} \geq .752001$.

6 Nash equilibria

It is natural to ask about the relationship between pro…tability as discussed above and Nash equilibrium. Consider, for instance, a location game on the unit interval $[0;1]$ with consumers distributed uniformly, played by several independent …rms and one player who controls a set of …rms. Can we …nd location strategies for the independent …rms so that these strategies together with the pro…table collusion strategy identi…ed above comprise a Nash equilibrium?

It is possible to answer this question in some special cases and we shall do this below. However, in general the problem is very di¢cult, perhaps intractable.

The di¢culty arises from two sources. The …rst thing one discovers when working on the problem is that a Nash equilibrium requires mixed strategies for the independent …rms. Unfortunately, due to the computational complexities, little is known about mixed strategy equilibria for location games on the unit interval. Shaked (1982) constructed a mixed strategy Nash equilibrium for three firms locating on $[0,1]$, and there are nonconstructive existence theorems by Dasgupta and Maskin (1986) and Simon (1987). Second, the difficulty in finding mixed strategy equilibria is compounded when one player controls m locations, due to the added computational complexity.

Prospects are even bleaker for location games with collusion in dimensions 2 and higher. Up to now, nothing has been published on location games in dimension 3 or above, and little on solutions for location games in dimension 2. Shaked (1975) showed that there are no pure-strategy Nash equilibria for a wide variety of 3-player location games in the plane; Okabe and Aoyagi (1993) proved that an infinite square array of firms in the plane is a Nash equilibrium for a uniform distribution of consumers, and Knoblauch (1997) catalogued all 3-player equilibria on the 2-sphere when consumers are distributed uniformly.

In summary, the difficulty of finding mixed strategy equilibria for location games translates into difficulty for our problem–finding equilibria for location games with collusion. It seems reasonable that firms that engage in games that game theorists are unable to solve should choose rather conservatively when it comes to making big decisions such as decisions about colluding or merging with others. We have therefore proposed profitability as a conservative criterion to be used by firms faced with collusion decisions or merger proposals.

Nevertheless, the following proposition answers the question posed at the beginning of this section in the affirmative in the special case that the number of locations controlled by the "big" player is an integral multiple of the number of independent firms.

Proposition 6 Let G be a location game in which consumers are distributed uniformly on $[0,1]$ with density 1, players $1,2,\ldots,n-m$, are independent firms, player $n-m+1$ controls m locations and $m = a(n-m)$ where a is a positive integer greater than 1. Let s_{n-m+1} be player $n-m+1$'s strategy from Section 3.1, which picks locations at each element of the set $\{k,3k,5k,\ldots,1-k\}$, where $k = 1/2m$. For $i = 1,2,\ldots,n-m$, let s_i be player i's strategy that assigns probability $\frac{1}{a}$ to each of the a points $(2ai-2a+1)/2m$, $(2ai-2a+3)/2m,\ldots,(2ai-1)/2m$.[5] Then $(s_1,s_2,\ldots,s_{n-m},s_{n-m+1})$

[5] To illustrate, consider the case in which there is only one independent player (i.e., m = a = n-1). In this case the independent player chooses each of the m equidistant locations chosen by the firm controlling the coalition with probabilty 1/m. Alternatively, consider the special case of n = 6 and m = 4 (a=2), i.e. there are two independent players and one player controlling a 4-firm coalition. In this case the coalition firms will occupy locations 1/8, 3/8, 5/8, and 7/8 whereas the first (second) independent player chooses locations 1/8 and 3/8 (5/8 and 7/8) with probability 1/2 each.

is a Nash equilibrium of G.

Proof Fix $i \in \{1, 2, \ldots, n-m\}$. Let H be a game like G but with only two players, so that 1 is an independent firm and 2 controls m locations. For $x \in [0,1]$,

$$\pi_i^G(s_1, \ldots, s_{i-1}, x, s_{i+1}, \ldots, s_{n-m+1}) \leq \pi_1^H(x, s_{n-m+1}) \leq 1/2m = \pi_i^G(s_1, s_2, \ldots, s_{n-m+1}).$$

It remains to show that player $n-m+1$ can't improve his payoff by a unilateral strategy change.

Let t_{n-m+1} be any pure strategy of player $n-m+1$. Let K be a game like G but with two players each of whom controls m locations. Then by the definitions of s_i,

$$\begin{aligned} & \sum_{i=1}^{n-m} \pi_i^G(s_1, s_2, \ldots, s_{n-m}, t_{n-m+1}) \\ = \; & \left(\frac{1}{a}\right) \sum_{i=1}^{n-m} \sum_{j=1}^{a} \pi_i^G(s_1, \ldots, s_{i-1}, (2ai-2a+2j-1)/2m, s_{i+1}, \ldots, s_{n-m}, t_{n-m+1}) \\ \geq \; & \pi_1^K(s_{n-m+1}, t_{n-m+1})/a. \end{aligned}$$

The inequality follows from the fact that any consumer (or fraction of a consumer) awarded to the first player in game K will contribute to one of the summands on the left side of the inequality. For example, suppose t_{n-m+1} assigns three locations to k, two locations to $5k + \frac{1}{2}$ and no location to any point between. How does the consumer interval $(2k, 3k)$ contribute to the two sides of the inequality? Player 1 in game K wins all of $(2k, 3k)$. Player 1 in game G wins all of $(2k, 3k)$ in the summand $\pi_i^G(3k, s_2, \ldots, s_{n-m}, t_{n-m+1})$ and one quarter of $(2k, 3k)$ in the summand $\pi_i^G(k, s_2, \ldots, s_{n-m}, t_{n-m+1})$.

Next $\pi_1^K(s_{n-m+1}, t_{n-m+1}) \geq (3m-n)/2m = 1/2$ by the profitability argument in Section 3.1. Combining the above inequalities,

$$\sum_{i=1}^{n-m} \pi_i^G(s_1, s_2, \ldots, s_{n-m}, t_{n-m+1}) \geq 1/2a = (n-m)/2m$$

Therefore $\pi_{n-m+1}^G(s_1, s_2, \ldots, s_{n-m}, t_{n-m+1}) \leq 1 - (n-m)/2m = (3m-n)/2m \leq$ $\pi_{n-m+1}^G(s_1, s_2, \ldots, s_{n-m}, s_{n-m+1})$ with the last inequality following again from the profitability argument of Section 3.1. □

7 Discussion

We …nd that collusion in location games only pays if the set of colluders is larger than the set of non–colluding competitors. Bilateral collusion, for example, can only pay if there are no more than three competitors. This result is based on an approach which relies on rather weak rationality requirements. It assumes that players discussing some binding agreements to collude will only go ahead if they can guarantee themselves a payo¤ better than the payo¤ expected "behind the veil of ignorance".

This maxmin approach prescribes that colluding players should spread themselves out, making players outside the colluding set indi¤erent between locations. This seems to be rather intuitive: One would expect that two colluding supermarkets (or supermarkets belonging to the same chain) locate in di¤erent parts of one city to avoid cannibalization. For a special case of competition on the unit interval, we show that the maxmin strategy is also used in a non-cooperative equilibrium.

Our results may have implications for the topic of mergers in markets with (pure) spatial competition as an example of which competition among big book retailers (where price competition is extremely limited) may serve. As merger in the traditional sense (see Salant, Switzer, and Reynolds 1983) where …rms simply "disappear" never pays in such location games, merger can only be pro…table if the merging units are kept as separate units which are governed by central headquarters. This is identical to the case of collusion analysed above. However, the analysis reveals that with this kind of competition only "mega mergers" are likely to occur.[6]

References

[1] Aoyagi, M., and A. Okabe (1993): Spatial competition of …rms in a two-dimensional bounded market, Regional Science and Urban Economics 23(2), 259-89.

[2] Dasgupta, P., and E. Maskin (1986a): The existence of equilibrium in discontinuous economic games, I: Theory, Review of Economic Studies 53, 1-26.

[6] Concerning the market for books such a mega merger has recently occurred in the UK where Waterstone's took over Dillon's. And, interestingly, the new Waterstone's branches in London are pretty much "spread out." In particular, Waterstone's two ‡agship stores are not at Charing Cross Road, the traditional spot for large book stores but rather "to the left and to the right" of the competitors' big stores, namely at Picadilly and UCL.

[3] Dasgupta, P., and E. Maskin (1986b): The existence of equilibrium in discontinuous economic games, II: Applications, Review of Economic Studies 53, 27-41.

[4] Hotelling, H. (1929): Stability in competition, Economic Journal 39, 41-57.

[5] Knoblauch, V. (1997): A pure strategy Nash equillibrium for a three-…rm location game on a two-dimensional set, Location Science 4, 247-250.

[6] Knoblauch, V. (2002): An easy proof that a square lattice is an equilibrium for spatial competition in the plane, Journal of Urban Economics 51, 46-53.

[7] Salant, S.W., S. Switzer, and R.J. Reynolds (1983): Losses from horizontal merger: The e¤ects of an exogenous change in industry structure on Cournot-Nash equilibrium, Quarterly Journal of Economics 98, 185-199.

[8] Shaked, A. (1975): Non-existence of equilibrium for the two-dimensional three-…rms location problem, Review of Economic Studies 42(1), 51-56.

[9] Shaked, A. (1982): Existence and computation of a mixed strategy equilibrium for 3-…rms location problem, Journal of Industrial Economics 31, 93-96.

[10] Simon, L. (1987): Games with discontinuous payo¤s, Review of Economic Studies 54, 569-597.